NTNU | Norwegian University of Science and Technology

Functors and subcategories of *n*-exangulated categories

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Plan of talk

- A brief introduction to n-exangulated categories
- Subcategories of n-exangulated categories
  - *The Grothendieck group of an n-exangulated category*, Applied Categorical Structures (2021)
- Functors in higher homological algebra
  - Joint with R. Bennett-Tennenhaus, M. H. Sandøy and A. Shah: *Structure-preserving functors between higher exangulated categories*, in progress

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Historical background

- Abelian categories (Buchsbaum 1955, Grothendieck 1957)
- Exact categories (Quillen 1973)
- Triangulated categories (Verdier 1996, Puppe 1962)

Distinguished class of 3-term sequences

$$0 \to A \to B \to C \to 0$$
$$A \to B \to C \to \Sigma A$$

- Extriangulated categories (Nakaoka–Palu 2019)

Historical background - higher homological algebra

- n-abelian and n-exact categories (Jasso 2016)
- $(n+2)$-angulated categories (Geiss–Keller–Oppermann 2013)

Distinguished class of $(n+2)$-term sequences

$$0 \to X^0 \to X^1 \to \cdots \to X^n \to X^{n+1} \to 0$$

$$X^0 \to X^1 \to \cdots \to X^n \to X^{n+1} \to \Sigma X^0$$

- n-exangulated categories (Herschend–Liu–Nakaoka 2021)

What is an n-exangulated category?

An n-exangulated category is a triplet $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ consisting of

- an additive category $\mathcal{C}$
- a biadditive functor $\mathbb{E}\colon\ \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathsf{Ab}$
- an exact realization $\mathfrak{s}$

Given $\delta \in \mathbb{E}(C, A)$, we have

$$\mathfrak{s}(\delta) = [\ \underbrace{A \to X^1 \to \cdots \to X^n \to C}_{\textit{conflation}}\ ].$$

What is an n-exangulated category?

Axioms

The triplet $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is n-exangulated if it satisfies certain axioms.

(EA1) The class of inflations in $\mathcal{C}$ is closed under composition. Dually, the class of deflations in $\mathcal{C}$ is closed under composition.

Proposition (Herschend–Liu–Nakaoka)

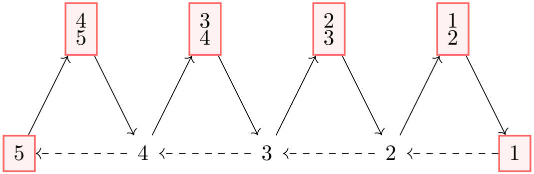
A triplet $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ as above is 1-exangulated if and only if it is an extriangulated category.

Examples

Consider the path algebra of the quiver

$$1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4 \longrightarrow 5$$

modulo all paths of length 2. The triplet $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is 4-exangulated:



Examples

Let $(\mathcal{C}, \Sigma, \pentagon)$ be an $(n+2)$-angulated category. Then $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is n-exangulated with

- $\mathbb{E} = \mathrm{Hom}_{\mathcal{C}}(-, \Sigma -) \colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathrm{Ab}$
- For $\delta \in \mathbb{E}(C, A)$, we have

$$\mathfrak{s}(\delta) = [A \longrightarrow X^1 \longrightarrow \cdots \longrightarrow X^n \longrightarrow C \overset{\delta}{\dashrightarrow} \Sigma A].$$

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The Grothendieck group of an n-exangulated category

From now, let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an essentially small n-exangulated category.

The isomorphism class of an object $X \in \mathcal{C}$ is denoted by $\langle X \rangle$. The free abelian group generated by such isomorphism classes is denoted by $\mathcal{F}(\mathcal{C})$.

Given a conflation

$$X^{\bullet}\colon X^0 \to X^1 \to \cdots \to X^n \to X^{n+1}$$

in $\mathcal{C}$, we write

$$\chi(X^{\bullet}) = \langle X^0 \rangle - \langle X^1 \rangle + \cdots + (-1)^{n+1} \langle X^{n+1} \rangle.$$

The Grothendieck group of an n-exangulated category

Definition (H.)

The *Grothendieck group* of $\mathcal{C}$ is the quotient $K_0(\mathcal{C}) = \mathcal{F}(\mathcal{C})/\mathcal{R}(\mathcal{C})$, where $\mathcal{R}(\mathcal{C})$ is the subgroup generated by the subset

$$\{\chi(X^\bullet) \mid X^\bullet \text{ is a conflation in } \mathcal{C}\} \text{ if } n \text{ is odd and}$$
$$\{\langle 0\rangle\}\cup\{\chi(X^\bullet) \mid X^\bullet \text{ is a conflation in } \mathcal{C}\} \text{ if } n \text{ is even.}$$

Classification of subcategories

Theorem (Thomason)

Let $\mathcal{T}$ be a triangulated category. There is a one-to-one correspondence

$$\{\text{subgroups of } K_0(\mathcal{T})\} \rightleftarrows \{\text{dense triangulated subcategories of } \mathcal{T}\}\,.$$

Classification of subcategories

Theorem (Thomason)

Let $\mathcal{T}$ be a triangulated category. There is a one-to-one correspondence

$$\{\text{subgroups of } K_0(\mathcal{T})\} \rightleftarrows \{\text{dense}^* \text{ triangulated subcategories of } \mathcal{T}\}.$$

(*) A subcategory $\mathcal{S} \subseteq \mathcal{T}$ is *dense* if every object in $\mathcal{T}$ is a direct summand of an object in $\mathcal{S}$.

Classification of subcategories

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Let $\mathcal{T}$ be a triangulated category. There is a one-to-one correspondence

$$\{\text{subgroups of } K_0(\mathcal{T})\} \rightleftarrows \{\text{dense triangulated subcategories of } \mathcal{T}\}.$$

Theorem (H.)

Let $\mathcal{C}$ be an n-exangulated category with n odd. Given an n-(co)generator $\mathcal{G}$ of $\mathcal{C}$, there is a one-to-one correspondence

$$\left\{\begin{matrix}\text{subgroups of } K_0(\mathcal{C}) \\ \text{containing the image of } \mathcal{G}\end{matrix}\right\} \rightleftarrows \left\{\begin{matrix}\text{dense complete subcategories} \\ \text{of } \mathcal{C} \text{ containing } \mathcal{G}\end{matrix}\right\}.$$

Classification of subcategories

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(*) A subcategory $\mathcal{G} \subseteq \mathcal{C}$ is called an *n-generator* of $\mathcal{C}$ if for each object A in $\mathcal{C}$, there exists a conflation

$$A' \to G^1 \to \cdots \to G^n \to A$$

in $\mathcal{C}$ with G^i in $\mathcal{G}$ for $i = 1, \ldots, n$.

Classification of subcategories

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$$\left\{\begin{array}{l}\text{subgroups of } K_0(\mathcal{C}) \\ \text{containing the image of } \mathcal{G}\end{array}\right\} \rightleftarrows \left\{\begin{array}{l}\text{dense complete}^{**} \text{ subcategories} \\ \text{of } \mathcal{C} \text{ containing } \mathcal{G}\end{array}\right\}.$$

(**) A subcategory $\mathcal{S} \subseteq \mathcal{C}$ is *complete* if given any conflation in $\mathcal{C}$ with $n+1$ of its objects in $\mathcal{S}$, also the last object has to be in $\mathcal{S}$.

Classification of subcategories

Theorem (H.)

Let $\mathcal{C}$ be an n-exangulated category with n odd. Given an n-(co)generator $\mathcal{G}$ of $\mathcal{C}$, there is a one-to-one correspondence

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Question

Can we say more about the structure of these subcategories?

n-exangulated functors

Let $(\mathcal{C}_1, \mathbb{E}_1, \mathfrak{s}_1)$ and $(\mathcal{C}_2, \mathbb{E}_2, \mathfrak{s}_2)$ be n-exangulated categories.

Question

When is an additive functor $F\colon \mathcal{C}_1 \longrightarrow \mathcal{C}_2$ an *n-exangulated functor*?

$$\begin{array}{ccc} \mathcal{C}_1^{\mathrm{op}} \times \mathcal{C}_1 & \xrightarrow{\mathbb{E}_1} & \mathrm{Ab} \\ {\scriptstyle F^{\mathrm{op}}\times F}\downarrow & \overset{\eta}{\swarrow\!\!\swarrow} & \downarrow{\scriptstyle Id} \\ \mathcal{C}_2^{\mathrm{op}} \times \mathcal{C}_2 & \xrightarrow{\mathbb{E}_2} & \mathrm{Ab} \end{array}$$

$$\begin{array}{rccc} {}_A\eta_C\colon & \mathbb{E}_1(C, A) & \longrightarrow & \mathbb{E}_2(FC, FA) \\ & \delta & \longmapsto & {}_A\eta_C(\delta) \end{array}$$

Answer (Bennett-Tennenhaus–Shah)

If there is a natural transformation $\eta\colon \mathbb{E}_1 \longrightarrow \mathbb{E}_2(F^{\mathrm{op}}-, F-)$ such that

$$\mathfrak{s}_1(\delta) = [X^\bullet] \implies \mathfrak{s}_2(\eta(\delta)) = [FX^\bullet].$$

n-exangulated subcategories

We call η an *inclusion* if

$$ {}_A\eta_C \colon \mathbb{E}_1(C, A) \longrightarrow \mathbb{E}_2(FC, FA) $$

is an inclusion of abelian groups for every pair of objects A and C.

Definition (H.)

A subcategory $\mathcal{S} \subseteq \mathcal{C}$ is an *n-exangulated subcategory* of $\mathcal{C}$ if $\mathcal{S}$ carries an n-exangulated structure for which the inclusion functor is n-exangulated and the associated natural transformation is an inclusion.

n-exangulated subcategories

Example (Herschend–Liu–Nakaoka)

Let $\mathcal{S} \subseteq \mathcal{C}$ be extension-closed. If $(\mathcal{S}, \mathbb{E}|_{\mathcal{S}^{\mathrm{op}} \times \mathcal{S}}, \mathfrak{s}')$ satisfies (EA1), then it is an n-exangulated subcategory.

Theorem (August–H.–Jacobsen–Kvamme–Palu–Treffinger)

Every n-torsion class of an n-abelian category is extension-closed.

n-exangulated subcategories

Open problem

Is (EA1) necessarily satisfied for an extension-closed subcategory of $\mathcal{C}$?

Proposition (H.)

If an additive subcategory $\mathcal{S} \subseteq \mathcal{C}$ is dense and complete, it carries the structure of an n-exangulated subcategory of $\mathcal{C}$.

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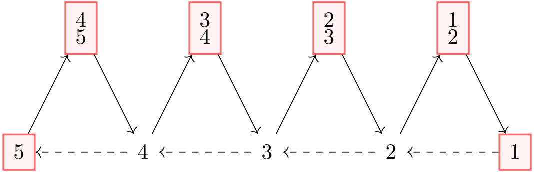
Motivation

Let Λ denote the path algebra of the quiver

$$1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4 \longrightarrow 5$$

modulo all paths of length 2.



$$\mathcal{C} \hookrightarrow \text{mod}\,\Lambda$$

conflation $\longmapsto$ 4 conflations

(n,d)-exangulated functors

The $(4,1)$-exangulated functor $\mathcal{C} \hookrightarrow \operatorname{mod}\Lambda$ sends

$$\text{conflation} \longmapsto 4 \text{ conflations.}$$

Let $\mathcal{C}_1$ be an n-exangulated and $\mathcal{C}_2$ a d-exangulated category for $n = kd$. An (n,d)*-exangulated functor* $\mathcal{C}_1 \longrightarrow \mathcal{C}_2$ sends

$$\text{conflation} \longmapsto k \text{ conflations.}$$

Examples of (n, d)-exangulated functors

Consider the path algebra of the quiver

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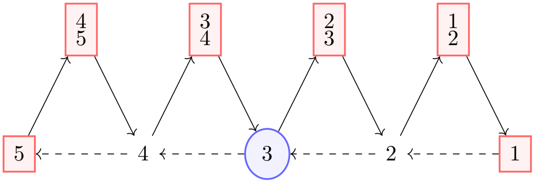
modulo all paths of length 2.



We have a $(4, 2)$-exangulated functor

$$\mathcal{C} \hookrightarrow \mathcal{D}$$

$$\text{conflation} \longmapsto 2 \text{ conflations}$$

Examples of (n, d)-exangulated functors

Examples arising from:

- Reduced structures (Herschend–Liu–Nakaoka)
- Higher Nakayama algebras (Jasso–Külshammer)
- Gluing systems of representation-directed algebras (Vaso)

Thank you for your attention :)

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