Canonical join and meet representations in lattices of torsion classes

Eric J. Hanson

NTNU

based on joint works with Emily Barnard and Kiyoshi Igusa

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Outline

1. Canonical join and meet representations
2. Lattices of torsion classes
3. Canonical collections
4. Preprojective algebras of type A_n
5. Patterns and generalizations

Canonical join and meet representations

Definition

Let L be a finite lattice, and let $x \in L$.

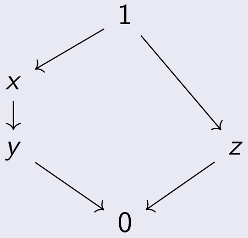
1. Let $S \subseteq L$. We say $\bigvee S$ is a *join representation* of x if $\bigvee S = x$.
2. A join representation $\bigvee S$ of x is *irredundant* if $\bigvee S' \lneq \bigvee S$ for all $S' \subsetneq S$.
3. The set of join representations of x forms a poset under the relation $\bigvee S \preceq \bigvee S'$ if for all $y \in S$ there exists $y' \in S'$ with $y \leq y'$.
4. If the poset in (3) contains a unique minimal element, we call this element the *canonical join representation* of x.

The definition of a canonical meet representation is analogous.

Canonical join and meet representations

Example

Let L be the following lattice:

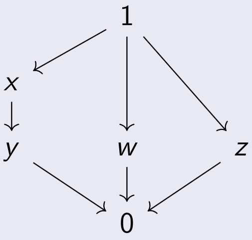


- There is a canonical join representation $1 = y \vee z$.
- There is a canonical meet representation $0 = x \wedge z$.
- All other canonical join representations are of the form $w = \bigvee\{w\}$ and $w = \bigwedge\{w\}$.

Canonical join and meet representations

Example

Let L be the following lattice:



Not every element has a canonical join representation. There are 3 minimal join representations of 1:

$$1 = y \vee w \qquad 1 = y \vee z \qquad 1 = w \vee z$$

Algebraic setting

- Let Λ be a finite-dimensional algebra over an arbitrary field K.
- Let modΛ be the category of finitely-generated (right) Λ-modules.
- A module $M \in \text{mod}\Lambda$ is called a *brick* if $\mathrm{End}(M)$ is a division algebra (over K).
- A set of bricks $\mathcal{X}$ is called a *semibrick* if $\mathrm{Hom}(M, N) = 0 = \mathrm{Hom}(N, M)$ for all $M \neq N \in \mathcal{X}$.

Algebraic Setting

Running Example

Consider the quiver

$$Q: \qquad 1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2$$

and let $A = KQ/(\alpha\beta, \beta\alpha)$.

As a vector space, $A = K\langle e_1, e_2, \alpha, \beta\rangle$. The only nonzero multiplications of generators are:

$$e_1 \cdot \alpha = \alpha = \alpha \cdot e_2 \qquad e_2 \cdot \beta = \beta = \beta \cdot e_1$$

Algebraic Setup

Running Example

Every nonzero module in mod$A$ is a direct sum of copies of four bricks:

$$S_1 = \; K \underset{0}{\overset{0}{\rightleftarrows}} 0$$

$$S_2 = \; 0 \underset{0}{\overset{0}{\rightleftarrows}} K$$

$$P_1 = \; K \underset{0}{\overset{1}{\rightleftarrows}} K$$

$$P_2 = \; K \underset{1}{\overset{0}{\rightleftarrows}} K$$

Moreover, there are exact sequences

$$S_1 \hookrightarrow P_2 \twoheadrightarrow S_2 \qquad S_2 \hookrightarrow P_1 \twoheadrightarrow S_1$$

Torsion Classes

Definition

Let $\mathcal{T}, \mathcal{F}$ be (full) subcategories of $\text{mod}\Lambda$.

1. The pair $(\mathcal{T}, \mathcal{F})$ is a *torsion pair* if $\mathcal{T} = {}^{\perp}\mathcal{F}$ and $\mathcal{F} = \mathcal{T}^{\perp}$, i.e.:

$$\begin{aligned} \mathcal{T} &= \{M \in \text{mod}\Lambda \mid \forall N \in \mathcal{F} : \mathrm{Hom}(M, N) = 0\} \\ \mathcal{F} &= \{N \in \text{mod}\Lambda \mid \forall M \in \mathcal{T} : \mathrm{Hom}(M, N) = 0\} \end{aligned}$$

2. If $(\mathcal{T}, \mathcal{F})$ is a torsion pair, then $\mathcal{T}$ is called a *torsion class* and $\mathcal{F}$ is called a *torsion-free class*.

- The motivating examples are torsion groups (every element has finite order) and torsion-free groups (every non-identity element has infinite order).
- Other examples are $(\text{mod}\Lambda, 0)$ and $(0, \text{mod}\Lambda)$.

Torsion Classes

Torsion classes and torsion-free classes can also be characterized internally:

Proposition

Let $\mathcal{C} \subseteq \text{mod}\Lambda$ be a full subcategory which is closed under isomorphisms. Then

1. *$(\mathcal{C}, \mathcal{C}^{\perp})$ is a torsion pair if and only if $\mathcal{C}$ is closed under extensions and quotients.*
2. *$({}^{\perp}\mathcal{C}, \mathcal{C})$ is a torsion pair if and only if $\mathcal{C}$ is closed under extensions and subobjects.*

- The torsion classes (resp. torsion-free classes) of modΛ form a lattice under containment [IRTT15]. These lattices are anti-isomorphic to one another.
- We assume the number of torsion classes in modΛ is finite (that is, Λ is τ-tilting finite [DIJ19]).

Torsion Classes

Running Example

Recall that $Q = \; 1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2$ and $A = KQ/(\alpha\beta, \beta\alpha)$. Then the lattice of torsion classes is:

$$
\begin{array}{ccccc}
 & \swarrow & \mathrm{mod}\Lambda & \searrow & \\
\mathrm{add}(P_1, S_1) & & & & \mathrm{add}(P_2, S_2) \\
\downarrow & & & & \downarrow \\
\mathrm{add}(S_1) & & & & \mathrm{add}(S_2) \\
 & \searrow & 0 & \swarrow &
\end{array}
$$

Minimal extending and co-extending modules

Definition-Theorem (Barnard-Carroll-Zhu '19)

Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair.

1. A brick $M \in \text{mod}\Lambda$ is called a *minimal extending module* for $\mathcal{T}$ if there is a cover relation $\mathcal{T} \lessdot \text{Filt}(\mathcal{T} \cup \{M\})$.
2. A brick $M \in \text{mod}\Lambda$ is called a *minimal co-extending module* for $\mathcal{F}$ if there exists a cover relation $\mathcal{F} \lessdot \text{Filt}(\mathcal{F} \cup \{M\})$. Equivalently, M is a minimal extending module for ${}^{\perp}\text{Filt}(\mathcal{F} \cup \{M\})$.

Minimal extending and co-extending modules

Theorem (Barnard-Carroll-Zhu '19)

Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair.

1. *Let M, N be minimal extending modules for $\mathcal{T}$. Then* $\mathrm{Filt}(\mathcal{T} \cup \{M\}) = \mathrm{Filt}(\mathcal{T} \cup \{N\})$ *if and only if* $M \cong N$.
2. *Let M, N be minimal co-extending modules for $\mathcal{F}$. Then* $\mathrm{Filt}(\mathcal{F} \cup \{M\}) = \mathrm{Filt}(\mathcal{F} \cup \{N\})$ *if and only if* $M \cong N$.

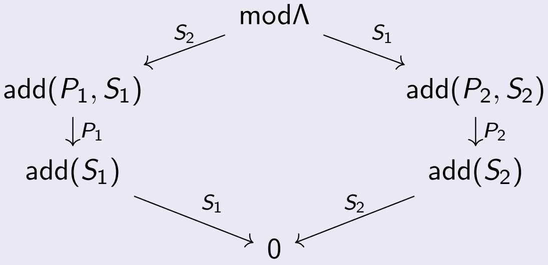
This theorem allows us to "label" the cover relation with bricks.

The notion of "brick labeling" is also developed independently by Demonet-Iyama-Reading-Reiten-Thomas [DIR$^+$], Asai [Asa20], and Brüstle-Smith-Treffinger [BST19].

Brick Labels

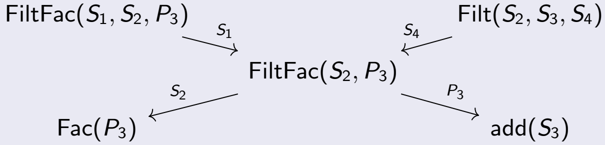
Running Example

Recall that $Q = \; 1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2$ and $A = KQ/(\alpha\beta, \beta\alpha)$. Then the torsion lattice is labeled as follows:



Example

The following is a piece of the lattice of torsion classes for the algebra $K(1 \to 2 \to 3 \to 4)$:



Observations:

1. Both $\{S_2, P_3\}$ and $\{S_1, S_4\}$ are semibricks.
2. FiltFac(S_2, P_3) can be described in terms of its minimal co-extending modules (the down labels).

Brick Labels

Theorem (Demonet-Iyama-Reading-Reiten-Thomas '17$^+$, Asai '20, Barnard-Carrol-Zhu '19)

Given a torsion pair $(\mathcal{T}, \mathcal{F})$, denote by $\mathcal{U}(T)$ the set of minimal extending modules for $\mathcal{T}$ and by $\mathcal{D}(T)$ the set of minimal co-extending modules for $\mathcal{F}$.

1. *The association $\mathcal{T} \mapsto \mathcal{U}(\mathcal{T})$ is a bijection between torsion classes and semibricks in* $\mathrm{mod}\Lambda$. *The inverse is $\mathcal{X} \mapsto {}^{\perp}\mathrm{FiltSub}(\mathcal{X})$.*
2. *The association $\mathcal{T} \mapsto \mathcal{D}(\mathcal{T})$ is a bijection between torsion classes and semibricks in* $\mathrm{mod}\Lambda$. *The inverse is $\mathcal{X} \mapsto \mathrm{FiltFac}(\mathcal{X})$.*
3. *The canonical meet representation of $\mathcal{T}$ is*
$$\bigwedge\{{}^{\perp}\mathrm{FiltSub}(M) | M \in \mathcal{U}(\mathcal{T})\}.$$
4. *The canonical join representation of $\mathcal{T}$ is*
$$\bigvee\{\mathrm{FiltFac}(M) | M \in \mathcal{D}(\mathcal{T})\}.$$

Semibricks describing torsion pairs

So far, we have seen that a set of modules corresponds to the canonical join representation of some torsion class (and the canonical meet representation of some other torsion class) if and only if it is a semibrick.

New question: When does a pair of sets of modules correspond to both a canonical join representation and a canonical meet representation of the same torsion class?

Precisely: Let $\mathcal{D}$ and $\mathcal{U}$ be semibricks. What are necessary and sufficient conditions for there to exist a torsion class $\mathcal{T}$ so that $\mathcal{D} = \mathcal{D}(\mathcal{T})$ and $\mathcal{U} = \mathcal{U}(\mathcal{T})$?

Semibricks describing torsion pairs

Theorem (Asai '20)

Let $\mathcal{D}$ and $\mathcal{U}$ be semibricks. Then there exists a torsion class $\mathcal{T}$ so that $\mathcal{D} = \mathcal{D}(\mathcal{T})$ and $\mathcal{U} = \mathcal{U}(\mathcal{T})$ if and only if $(\mathcal{D}, \mathcal{U})$ is a 2-simple minded collection. That is:

1. *For all $M \in \mathcal{D}$ and $N \in \mathcal{U}$, we have*
$$\mathrm{Hom}(M, N) = 0 = \mathrm{Ext}(M, N).$$
2. *$\mathcal{D} \cup \mathcal{U}$ satisfies a generating condition.*

Refined question: Let $\mathcal{D}$ and $\mathcal{U}$ be semibricks. What are necessary and sufficient conditions for there to exist a torsion class $\mathcal{T}$ so that $\mathcal{D} \subseteq \mathcal{D}(\mathcal{T})$ and $\mathcal{U} \subseteq \mathcal{U}(T)$?

We abbreviate this property by saying $(\mathcal{D}, \mathcal{U})$ is *contained in a canonical collection*.

Motivation

- One motivation for studying this question from the study of *picture groups* and *picture spaces*.
- The picture group of an algebra was first defined by Igusa-Todorov-Weyman [ITW] in the (rep.-finite) hereditary case and later generalized to τ-tilting finite algebras in [HI].
- It is a finitely presented group whose relations encode the structure of the lattice of torsion classes.
- The corresponding picture space is the classifying space of the *(τ)-cluster morphism category* [IT, BM19] of the algebra.

Motivation

Theorem (H.-Igusa '18^+)

Let Λ be a τ-tilting finite algebra. Then (assuming a technical hypothesis), the picture group and picture space have isomorphic (co-)homology if and only if the following are equivalent for all pairs of semibricks $(\mathcal{D}, \mathcal{U})$:

1. *$(\mathcal{D}, \mathcal{U})$ is contained in a canonical collection.*
2. *For all $M \in \mathcal{D}$ and $N \in \mathcal{U}$, the pair $(\{M\}, \{N\})$ is contained in a canonical collection.*

When this property holds, we say the algebra satisfies the *pairwise compatibility property*.

Necessary Conditions

The following comes from the definition of a 2-simple minded collection.

Proposition

Let $\mathcal{D}$ and $\mathcal{U}$ be semibricks. If $(\mathcal{D}, \mathcal{U})$ is contained in a canonical collection, then for all $M \in \mathcal{D}$ and $N \in \mathcal{U}$, we have

$$\mathrm{Hom}(M, N) = 0 = \mathrm{Ext}(M, N).$$

We will call this condition (C1). It is a pairwise condition!

For (representation-finite) hereditary algebras, (C1) is also sufficient. In particular, these algebras satisfy the pairwise compatibility property [IT].

Necessary Conditions

Proposition

Let $\mathcal{D}$ and $\mathcal{U}$ be semibricks. If $(\mathcal{D}, \mathcal{U})$ is contained in a canonical collection, then for all $M \in \mathcal{D}$, $N \in \mathcal{U}$, and $f : N \to M$, we have one of the following:

1. *f is the zero map.*
2. *f is injective.*
3. *f is surjective.*

We will call this condition (C2). It is a pairwise condition!

Condition (C2) can be proven two ways:

1. Using the definitions of minimal extending and co-extending modules and properties of torsion pairs.
2. Using the *mutation formula* for 2-simple minded collections [BY13, KY14].

Condition (C2)

Example

Let $Q = 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$ and let $\Lambda = KQ/(\alpha\beta)$. Then $(\{P_1\}, \{P_2\})$ satisfies (C1):

$$\mathrm{Hom}(P_1, P_2) = 0 = \mathrm{Ext}(P_1, P_2).$$

However, it does not satisfy (C2): there is a nonzero morphism $P_2 \to P_1$ with image S_2.

Theorem (H.-Igusa '18^{+})

Let $\mathcal{D}$ and $\mathcal{U}$ be semibricks over a Nakayama algebra (such as in the example). Then $(\mathcal{D}, \mathcal{U})$ is contained in a canonical collection if and only if it satisfies (C1) and (C2). In particular, Nakayama algebras satisfy the pairwise compatibility property,

More resolved cases

In [HI20], we give a necessary and sufficient conditions for a pair $(\mathcal{D}, \mathcal{U})$ to be contained in a canonical collection.

This condition essentially says that $(\mathcal{D}, \mathcal{U})$ satisfies (C1) and is *iteratively* compatible with the mutation formula for 2-simple minded collections. It allows us to prove the following.

Theorem (Barnard-H. '20$^+$)

Let W be a (simply-laced) finite Weyl group. Then the preprojective algebra of type W satisfies the pairwise compatibility property if and only if W is type A_1, A_2, or A_3.

Theorem (H.-Igusa '20)

Let $\Lambda = KQ/I$ be a τ-tilting finite gentle algebra and assume Q contains no loops or 2-cycles. Then Λ satisfies the pairwise compatibility property if and only if Q contains no vertex of degree larger than 2.

Preprojective algebras of type A_n

Recall that the Weyl group A_n is isomorphic to the symmetric group on $[n+1]$. We write elements of A_n in one-line notation.

Definition

Let $w \in A_n$.

1. A *descent* of w is a pair $(p,q) \in [n+1]^2$ so that $p > q$ and there exists $i \in [n]$ so that $p = w(i)$ and $q = w(i+1)$.
2. An *ascent* of w is a pair $(p,q) \in [n+1]^2$ so that $p < q$ and there exists $i \in [n]$ so that $p = w(i)$ and $q = w(i+1)$.

Example

Suppose $n = 3$ and $w = 213$. Then the w has a descent $(2,1)$ and an ascent $(1,3)$.

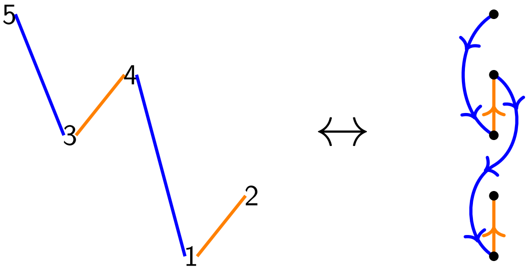
Preprojective algebras of type A_n

Descents and assents encode the canonical join and meet representations in the *weak order* on A_n. This order is isomorphic to the lattice of torsion classes of the corresponding preprojective algebra [Miz14].

Question: Let D and A be sets of pairs in $[n+1]^2$. Under what conditions does there exist a permutation $w \in A_n$ so that the tuples in D are descents and the tuples of A are ascents?

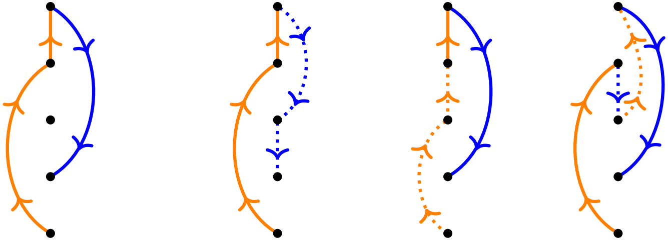
2-colored arc diagrams

Given $w \in A_n$, we visualize ascents and descents by graphing and connecting the points $(i, w(i))$. If we move this pattern into a straight line, we get a *2-colored arc diagram*.



- The definition of arc diagrams is due to Reading [Rea15]. They correspond to canonical join representations in the weak order.
- Barnard-Carrol-Zhu interpret arcs as bricks over the preprojective algebra of type A_n. With this interpretation, arc diagrams correspond to semibricks.

Type A_4



- There is no additional arc which can be added to the 2-colored arc diagram on the left.
- After deleting any one arc from the left diagram, the result can be "completed" to a 2-colored arc diagram with 4 arcs.
- This shows that the preprojective algebra of type A_4 does not satisfy the pairwise compatibility property.

Type A_n

Theorem (Barnard-H. '20$^+$)

Let $n \in \mathbb{N}$.

1. *Let L be a 2-colored arc diagram on $n+1$ nodes. Then there exists a permutation in A_n whose descents correspond to the blue arcs in L and whose ascents correspond to the orange arcs in L if and only if L contains n arcs.*
2. *Let $(\mathcal{D}, \mathcal{U})$ be a pair of semibricks over the preprojective algebra of type A_n. Then $(\mathcal{D}, \mathcal{U})$ is a canonical collection if and only if it satisfies (C1) and $|\mathcal{D}| + |\mathcal{U}| = n$.*

A Rank 3 Pattern Emerges

Conjecture

Let Λ be a τ-tilting finite algebra and let $\mathrm{rk}(\Lambda)$ be the number of simple modules in modΛ *(up to isomorphism). Then a pair of semibricks $(\mathcal{D},\mathcal{U})$ in* modΛ *is a canonical collection if and only if it satisfies (C1) and $|\mathcal{D}| + |\mathcal{U}| = \mathrm{rk}(\Lambda)$.*

Theorem (Barnard-H. '20$^+$)

The conjecture holds if $\mathrm{rk}(\Lambda) \leq 3$. In particular, if $\mathrm{rk}(\Lambda) \leq 3$, then Λ satisfies the pairwise compatibility property.

A Rank 3 pattern emerges

Theorem (Barnard-H.)

Let Λ be any τ-tilting finite algebra. Then the following are equivalent.

1. *Λ has the pairwise compatability property.*
2. *An arbitrary pair of semibricks $(\mathcal{D},\mathcal{U})$ satisfying $|\mathcal{D}| + |\mathcal{U}| = 3$ is contained in a canonical pair if and only if for all $M \in \mathcal{D}$ and $N \in \mathcal{U}$, the pair $(\{M\},\{N\})$ is contained in a canonical pair.*

Small pairs of semibricks

Let us now weaken the condition that $|\mathcal{D}| + |\mathcal{U}| = \mathrm{rk}(\Lambda)$.

Definition

Let $(\mathcal{D},\mathcal{U})$ be a pair of semibricks which satisfies (C1). Let $W \subseteq \mathrm{mod}\Lambda$ be the smallest subcategory satisfying the following:

1. W contains all of the bricks in $\mathcal{D} \cup \mathcal{U}$.
2. W is full and closed under isomorphisms.
3. W is closed under extensions, kernels, and cokernels.

Then we say $(\mathcal{D},\mathcal{U})$ is *small* if the number of objects which are simple in W (up to isomorphism) is equal to $|\mathcal{D}| + |\mathcal{U}|$.

Note: If $|\mathcal{D}| + |\mathcal{U}| = n$, then $(\mathcal{D},\mathcal{U})$ is necessary small.

Future work

Conjecture

Let Λ be a τ-tilting finite algebra and let $(\mathcal{D},\mathcal{U})$ be a pair of semibricks in modΛ. *Then $(\mathcal{D},\mathcal{U})$ is contained in a canonical collection if and only if it satisfies (C1) and is small.*

Conjecture

Let Λ be a τ-tilting finite algebra. Then the following are equivalent:

1. *Λ has the pairwise compatability property.*
2. *An arbitrary pair of semibricks $(\mathcal{D},\mathcal{U})$ satisfying $|\mathcal{D}| + |\mathcal{U}| = 3$ and (C1) is small if and only if for all $M \in \mathcal{D}$ and $N \in \mathcal{U}$, the pair $(\{M\},\{N\})$ is small.*

Thank you!!

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