The $\mathbf{g}$-representation type of concealed algebras and incidence algebras of posets

[arχiv:2407.17965]

Erlend Børve (Cologne)

Joint with
Jacob Grevstad (NTNU)
Endre Rundsveen (NTNU)

Representation theory – combinatorial aspects and applications to TDA

Trondheim, 5th December 2024

Let $\tau : \mathrm{mod}(\Lambda) \longrightarrow \mathrm{mod}(\Lambda)$ denote the Auslander–Reiten translation.

A pair (M,P) of fin. gen. left Λ-modules is a <u>τ-rigid pair</u> if

i) M is <u>τ-rigid</u>, i.e. $\mathrm{Hom}_\Lambda(M, \tau M) = 0$.

ii) P is projective and $\mathrm{Hom}_\Lambda(P, M) = 0$.

<u>Def</u> [Demonet – Iyama – Jasso]: We say that Λ is <u>g-finite</u> if the set

$T_1 = \mathrm{ind}\,\tau\text{-rigid pair}(\Lambda) := \{ (M,P) \mid (M,P)\ \tau\text{-rigid pair}(\Lambda) \text{ s.t. } |M|+|P|=1 \} / \mathrm{iso}$

is finite.

<u>Thm.</u> [Demonet – Iyama – Jasso]: Let $\Delta(\Lambda) = \{ \{ (M_i, P_i) \in T_1 \mid (\bigoplus_i M_i, \bigoplus_i P_i) \in \tau\text{-rigid pair}(\Lambda) \} \}$.

This is a pure $(n-1)$-dimensional abstract simplicial complex.

Let $K_0(\text{proj}\,\Lambda)_{\mathbb{R}} := K_0(\text{proj}\,\Lambda)\otimes\mathbb{R}$ be the real Grothendieck group

basis given by $\{[P_i] : P_i \text{ indec. projective}\}$.

$[P\oplus Q] = [P] + [Q]$.

For $(M,P)\in T_1$, define the <u>g-vector</u> $g_{(M,P)} \in K_0(\text{proj}\,\Lambda)_{\mathbb{R}}$ by

$$g_{(M,P)} := \begin{cases} [P_0^M] - [P_1^M] & \text{if } P\simeq 0, \text{ where } P_1^M \xrightarrow{p} P_0^M \text{ is a min. proj pres. of } M. \\ -[P] & \text{if } M\simeq 0 \end{cases}$$

For a general τ-rigid pair (N,Q) define the <u>cone</u>

$$\mathcal{C}_{(N,Q)} := \text{span}_{\geq 0}\{g_{(M,P)} \mid (M,P)\in T_1,\ M\oplus N, \text{ and } P\oplus Q\} \subseteq \mathbb{R}^n$$

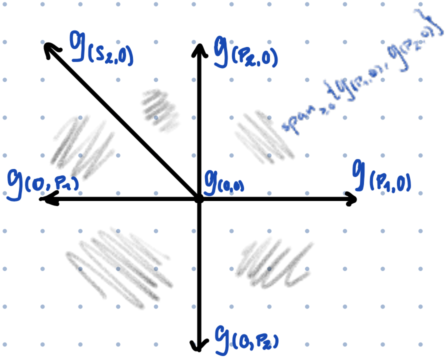
The <u>g-vector fan</u> of Λ is the collection $F_g(\Lambda)$ of these cones.

The g-vector fan $\mathcal{F}_g(\Lambda)$ is a fan:

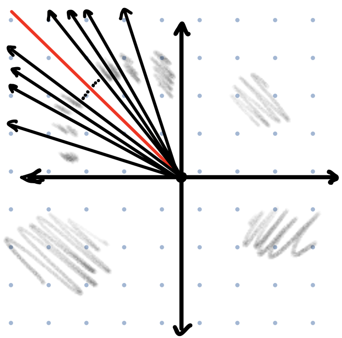
faces of $\mathcal{C}_{(N,Q)} = \{\mathcal{C}_{(M,P)} \mid M \triangleleft_{\oplus} N, \text{ and } P \triangleleft_{\oplus} Q\}$

$\mathcal{C}_{(N_1,Q_1)} \cap \mathcal{C}_{(N_2,Q_2)} = \mathcal{C}_{(N,Q)}$, (N,Q) largest common $\triangleleft_{\oplus}$

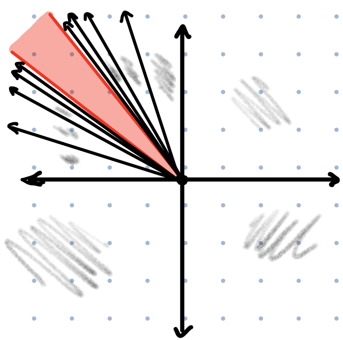
E.g. $k[1 \leftarrow 2]$ $\qquad$ $k[1 \leftleftarrows 2]$ $\qquad$ $k[1 \Lleftarrow 2]$



complete in $\mathbb{R}^n$



dense in $\mathbb{R}^n$



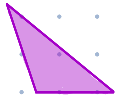
not dense in $\mathbb{R}^n$

$\overset{[\text{Asai}]}{\Longleftrightarrow}$ Λ g-finite

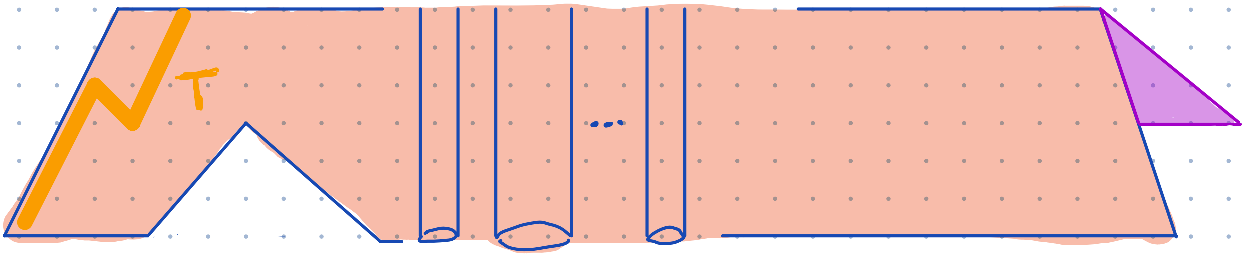
Q: rep.-infinite quiver.

Let $T \in \mathcal{P}$ be tilting (i.e. $\mathrm{Ext}^1_{kQ}(T,T)=0$, $|T|=n$),

let $\mathrm{Gen}(T) := \{X \in \mathrm{mod}(kQ) \mid \exists \text{ epi } T^{\oplus m} \twoheadrightarrow X\} =$,

and $T^{\perp} := \{X \in \mathrm{mod}(kQ) \mid \mathrm{Hom}_{kQ}(T,X) = 0\} =$ .

<u>Concealed algebra of type Q</u> $\Gamma := \mathrm{End}_{kQ}(T)$.



$\mathcal{P}_{\Gamma}$ post-proj. $\underbrace{}$ $\mathcal{R}_{\Gamma} = \mathcal{R}$ regular $\mathcal{I}_{\Gamma}$ pre-inj.

Def. L is a <u>contraction</u> of P if

$\exists$ epi. $P \xrightarrow{\gamma} L$ w/ connected fibres

↳ i.e $\forall x, y \in$ fibre $\exists\ x \leq \underbrace{z_1 \geq z_2 \leq \cdots}_{\in \text{fibre}} \leq y$

E.g.

$$\begin{tikzcd}[column sep=small] & a_1 \arrow[r, "\alpha"] & a_2 & \\ x \arrow[ur] \arrow[dr] & & & y \arrow[ul] \arrow[dl] \\ & b_2 & b_1 \arrow[l, "\beta"'] & \end{tikzcd}$$

contracts onto

$$\begin{tikzcd}[column sep=small] & a_{12} & \\ x \arrow[ur] \arrow[dr] & & y \arrow[ul] \arrow[dl] \\ & b_{12} & \end{tikzcd}$$

Def. A poset P is <u>minimally rep-infinite</u> if

i) $I(P)$ is rep.-infinite

ii) $\forall L \lneq P$; $I(L)$ rep.-finite

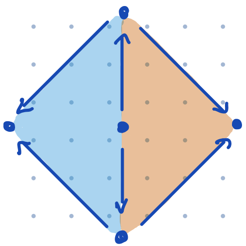
iii) $\forall$ proper contraction L : $I(L)$ is rep.-finite

A subposet $L \leq P$ is <u>convex</u> if $\left(\begin{array}{c} x \leq z \leq y \\ \text{with } x \in L,\ y \in L \end{array} \Rightarrow z \in L \right)$

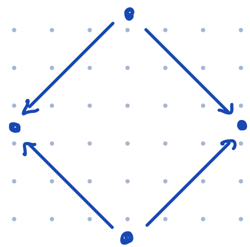
$I(L)$ becomes a <u>convex subcategory</u> of $I(P)$

i.e. $\left(\underset{\in \text{ ind proj } I(L)}{P_0} \to \underbrace{P_1 \to \cdots \to P_{m-1}}_{\in \text{ ind proj } I(P)} \to \underset{\in \text{ ind proj } I(L)}{P_m} \Rightarrow P_1, \ldots, P_{m-1} \in \text{ind proj } I(L) \right)$

Def: P is <u>simply connected</u> if its geometric realisation is.



is simply connected



is multiply connected