Reinhard F. Werner
Institut für Theoretische Physik
Leibniz Universität Hannover

Quantum Harmonic Analysis: Motivations and basic structure

Workshop
Quantum Harmonic Analysis and Applications to Operator Theory
Trondheim via Zoom

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Quantum harmonic analysis on phase space

R. Werner
Fachbereich Physik, Universität Osnabrück, Postfach 4469, D-4500 Osnabrück, West Germany

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133 citations

Physical Uniformities on the State Space of Nonrelativistic Quantum Mechanics

Reinhard Werner[1]

Received January 6, 1983

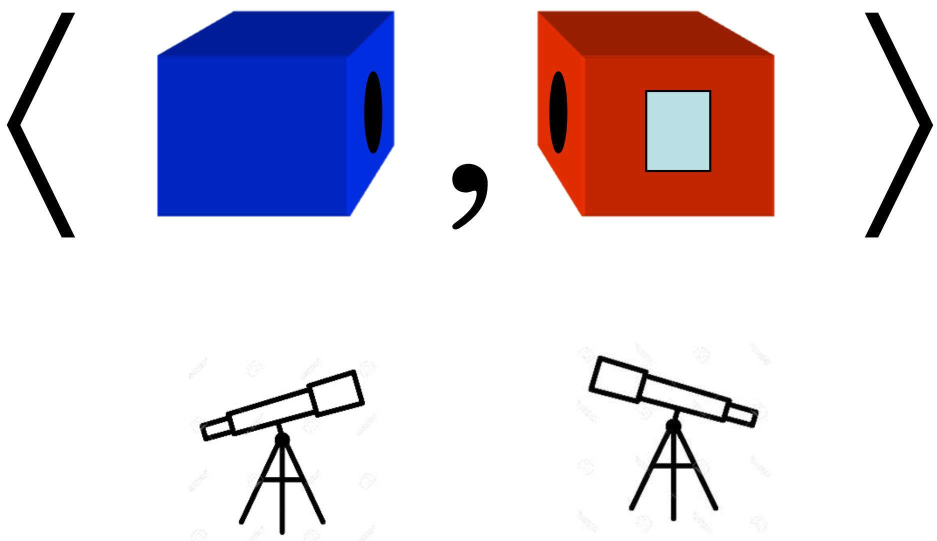
Found. Phys. **13** (1983) 859-881, birthday volume for Ludwig

Outline

- Statistical Dualities
- Phase Space Quantum Mechanics
- Convolutions & All That
- Correspondence

Statistical Dualities

and uniform structures

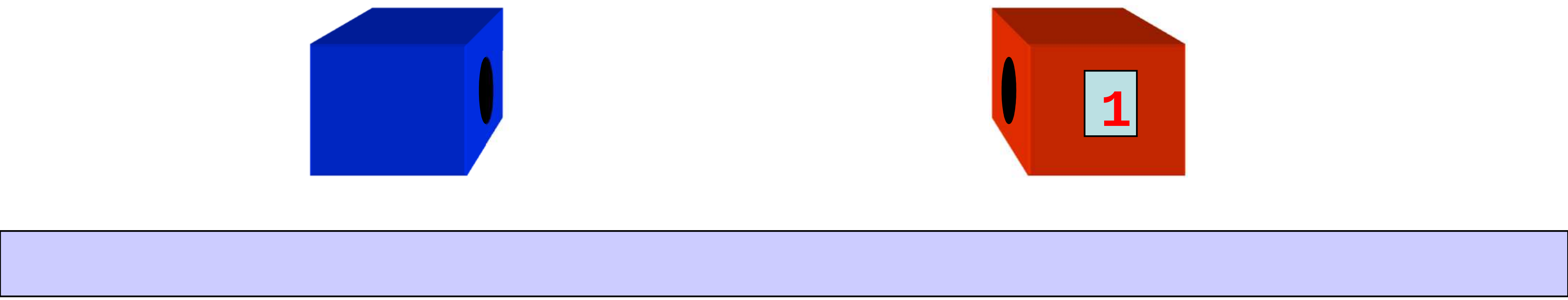


Quantum Mechanics and Probability

(in Statisical Interpretation)

preparation, state

measurement, observable



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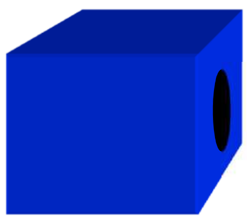
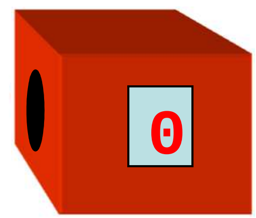


measurement, observable



1000100101100110101001100101110101000100100100110

$$p(\mathbf{1}) \approx \frac{22}{49} \qquad p(\mathbf{0}) \approx \frac{27}{49}$$

Theory **only** refers to such probabilities.

This point of view (=minimal statistical interpretation) was pushed and axiomatized by Günther Ludwig.

It works perfectly (although rarely acknowledged) as the underlying operational interpretation needed for Quantum Information Theory.

Axiomatics= „generalized probabilistic theories“, GPTs

One of Ludwig‘s themes: imprecision

Infinity must be tamed by Topology

We may be using wildly infinite constructions

„Physical distinguishability“ provides the grains of salt

Geometries over
real, rational, algebraic,
constructible, non-standard
numbers

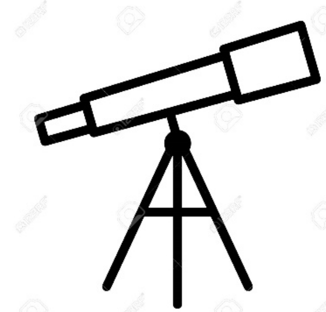
are physically indistinguishable

Notion of refinable distinguishing process

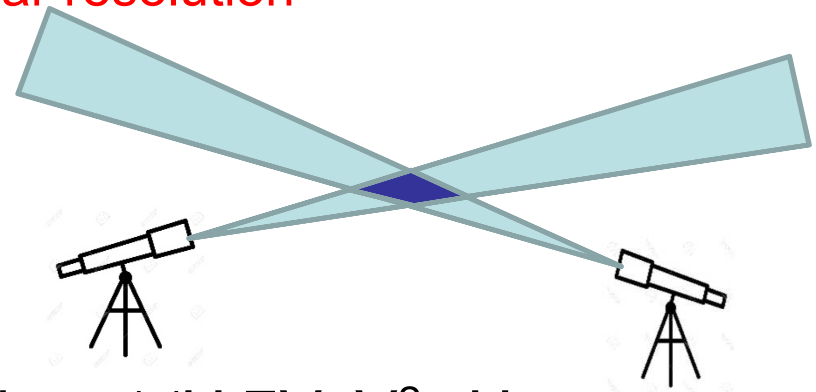
=

Uniform Structure (≈ non-parametric metrics)

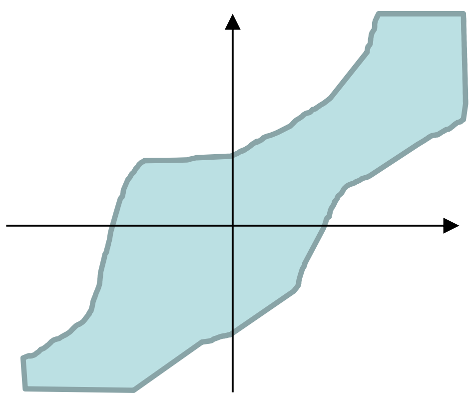
Example: Positional Astronomy



$\Rightarrow$ **Star catalogue**
based on observations of stars $\in X = \mathbf{R}^3$
from many observatories with finite angular resolution



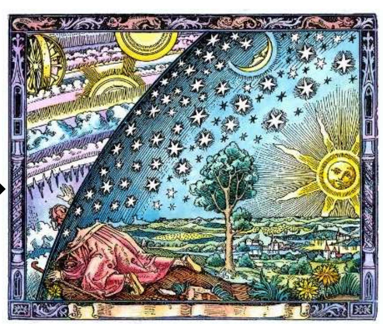
Each observatory has a neighborhood $U \subset X \times X$



refinement axiom: $\forall U\ \exists V$: $V^2 \subset U$

Uniform structure („uniformity“):
Natural home of the completion construction

Finite angular resolution



bdd uniformly cont. fcts

$\Leftrightarrow$

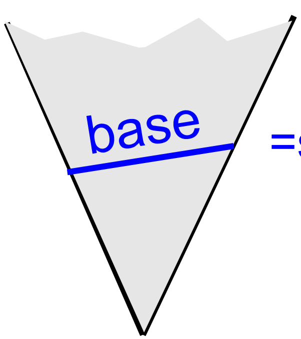
fcts extending to completion
compactification

What mathematical objects
are/should be used to represent these ?

preparation | measurement
state | observable

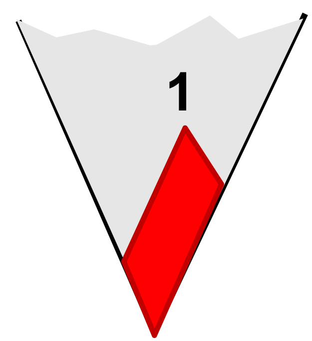
Ordered Banach Spaces

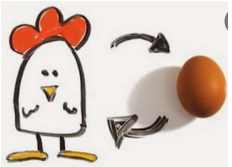
base normed | order unit



=state space

Positivity !





Observables assign outcome probabilities to states

States assign expectation values to observables

Which space is the dual?

Classical

states	observables
Meas(X)	$\mathcal{L}^{\infty}(X,\mu)$
$\mathcal{L}^{1}(X,\mu)$	$\mathcal{C}(X)$

Quantum

states	observables
$\mathcal{D}^*$	$\mathcal{B}(\mathcal{H})$
$\mathcal{T}(\mathcal{H})$	$\mathcal{D}$

Topology or
Measure Theory?

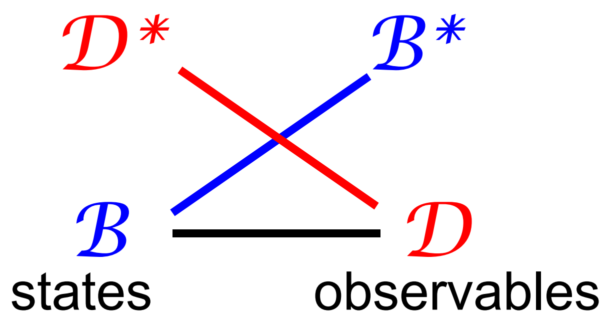
Günther Ludwig:
something is missing

VII. 8. The Problem of the Space $\mathcal{D}$ for Elementary Systems

Foundations of quantum mechanics, vol I (1983)

A more balanced view: have both $\langle \mathcal{B}, \mathcal{D} \rangle$

Think in terms of dualities of top. vector spaces:



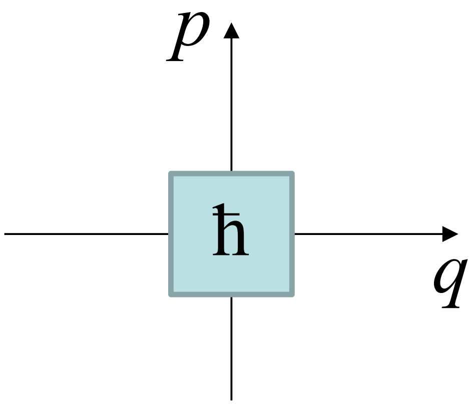
& bilinear form $\langle\ .\ ,\ .\ \rangle$
for probability

Weak topologies $\sigma(\mathcal{B}, \mathcal{D})$ and $\sigma(\mathcal{D}, \mathcal{B})$
describe “physical distinguishability“

Algebraically:
von Neumann algebra $\mathcal{B}^*$ with weak* dense C*- algebra $\mathcal{D}$

Phase Space

and quantum mechanics



Early history all in 1925/26:

Weyl:

writes to Born how to understand P&Q by commutation relations of 1-parameter groups

Born/Jordan/Heisenberg:

canonical commutation relations $[P,Q]=i\hbar$

Dirac:

c-numbers $\rightarrow$ q-numbers
Poisson brackets $\rightarrow$ commutators

Schrödinger:

inverts the classical limit (Hamilton-Jacobi $\rightarrow$ quantum)

von Neumann:

Hilbert space, density operators, and all the rest.

Phase Space ⟷ Quantum Mechanics

representation
of symmetry

Weyl operators

Galilei group:
space&velocity translations

$\Rightarrow$Unitary operators W(p,0) (momentum shift by p)
W(0,q) (spatial shift by q)

$W(p,0)W(0,q)=e^{ip\bullet q}\,W(0,q)W(p,0)$ canonical commutation relation

$W(p,q)= e^{ip\bullet q/2}\,W(0,q)W(p,0)$

No further degrees of freedom = {W(p,q)} irreducible
von Neumann: + continuity $\Rightarrow \mathcal{H}= \mathcal{L}^2(\mathbf{R}^3,dx)$, W(0,q)=shift, W(p,0)=multiply

Generalize: $\Xi=X^*\times X$

{positions}= loc compact abelian group X
{momenta}= dual group X^*

Physicist‘s intuitions:

Preperation uncertainty:

for no quantum state position and momentum distributions are both sharp

Joint measurement of P&Q:

($\rightarrow$Kiukas‘ talk)

possible, but with unavoidable errors (measurement uncertainty relations)

with proper quantitative notions & in general phase spaces:
measurement uncertainty=preparation uncertainty

Classical Limit of Dynamics

Poisson brackets / bound states/ scattering… see talk of Lauritz van Luijk

Generalized Weyl asymptotics (classical limit of thermodynamics):

one quantum state per phase space cell of size $(2\pi\hbar)^d$

Wigner functions

quantum states $\approx$ phase space probability density waiving positivity

Joint measurement of P&Q (state in 1983):

possible, but with unavoidable errors: Brian Davies, Alexander Holevo
I worked on this to get macroscopic observable for Boltzman statistical mechanics

Need to assign to every quantum state ρ ($\rho \geq 0$, tr $\rho=1$)
a probability distribution **m** on phase space.
Can be done covariantly: $\mathbf{m}(d\xi)=\mathrm{tr}(\rho\, W(\xi) D\, W(\xi)^*)\, d\xi$

The pun that got me started:
This works iff the operator valued Radon-Nikodym density D
is a density operator .

$\Rightarrow$ There is a binary operation here $(\rho, D) \rightarrow dm/d\xi \in L^1(\Xi, d\xi)$
operating via shift&average : convolution

Why correspondence theory? (→ Robert Fulsche's talk)

$$\begin{array}{c} \text{Quantum observable} \\ = \text{Hilbert space op} \end{array} \quad \underset{D*}{\overset{D*}{\rightleftarrows}} \quad \begin{array}{c} \text{Classical random variable} \\ = \text{phase space fct} \end{array}$$

depends on the choice of D

<u>except in settings of</u>

- D=canonized as Gaussian (Toeplitz op world)
- Wigner correspondence, wildly singular for uniform norms
- Classical limit

But

$$\begin{array}{c} \text{Translation inv.} \\ \text{subspace of ops} \end{array} \quad \underset{D*}{\overset{D*}{\rightleftarrows}} \quad \begin{array}{c} \text{Translation inv.} \\ \text{subspace of fcts} \end{array}$$

essentially independent of D

The Problem of the Space $\mathscr{D}$ for Elementary Systems is identical with its classical counterpart

Convolutions

and all that

$$*$$

Setting: $\Xi = X^* \times X$, X locally compact abelian

1983/84 papers are written for $X = \mathbf{R}^d$
Will track here what goes through ($\to$ BSc Thesis Niklas Galke 2020)
useful, e.g. for phase/number $X = \mathbf{Z}$, qubits $X = \{0,1\}$ and combinations

Von Neumann's uniqueness ✓
(via induced representations: spectral measure covariantly shifted)
$\Rightarrow$ $\mathcal{H} = \mathcal{L}^2(X, dx)$; Weyl operators act by shift& character multiplication

Write groups additively $\omega(p,q)$= bicharacter =exp(ip.q)
Weyl ops= $W(p,q) = \exp(i\, f(p,q))\, W(p,0) W(0,q)$ phase convention!
Commutation phase $W(\xi)\, W(\eta) = \sigma(\xi,\eta)\, W(\eta)\, W(\xi)$ independent of convention

Translations: $\alpha_\xi(A) = W(\xi)^* A\, W(\xi)$ for operators
$\alpha_\xi(f)\,(\eta) = f(\eta - \xi)$ for functions

Measure $d\xi$ is uniquely fixed by reciprocity of Haar measures
($\Rightarrow$ physicist's cell size $2\pi\hbar = 2\pi$)

Key Lemma: square integrability: $\langle \varphi, W(.)\,\psi \rangle \in \mathcal{L}^2(\Xi, d\xi)$

$$\| \langle \varphi, W(.)\,\psi \rangle \|_2^2 = \|\varphi\|^2 \, \|\psi\|^2$$

[You will hear more about integrability:
$\langle \varphi, W(.)\,\psi \rangle \in \mathcal{L}^1(\Xi, d\xi)$ if $\varphi, \psi \in$ Feichtinger algebra S_0]

Consequence: $\mathrm{tr}\, \rho_1\, \alpha_\bullet(\rho_2) \in \mathcal{L}^1(\Xi, d\xi)$, and $\int d\xi\, \mathrm{tr}\, \rho_1 \alpha_\xi(\rho_2) = (\mathrm{tr}\, \rho_1)\,(\mathrm{tr}\, \rho_2)$
for ρ_1, ρ_2 trace class.

tr $\leftrightarrow \int d\xi$, keep products for bilinear expressions; f,g= fcts, A,B= ops
reflection: $(\beta_- g)(\eta) = g(-\eta)$. $(U_- \psi)(x) = \psi(-x)$. $\beta_- A = U_- A U_-$

$$(f * g)(\xi) = \int d\eta \;\; f(\eta) g(\xi - \eta) = \int d\eta \;\; f(\eta)\,(\alpha_\eta g)(\xi) = \int d\eta \;\; f(\eta)\,(\alpha_\xi \beta_- g)(\eta)$$
$$f * A \quad = \qquad\qquad = \int d\eta \;\; f(\eta)\,(\alpha_\eta A)$$
$$A * g \quad = g * A$$
$$(A * B)(\xi) = \qquad\qquad = \mathrm{tr}\,(A\,(\alpha_\xi \beta_- B))$$

Then $\mathcal{L}^1(\Xi, d\xi) \oplus \mathcal{T}^1(\mathcal{H})$ becomes a **graded commutative Banach algebra**
trace class

Same formulas work, when one factor is from $\mathcal{L}^\infty(\Xi, d\xi) \oplus \mathcal{B}(\mathcal{H})$

Find the function representation $\mathcal{F}$!

Using $\mathcal{F}\,\alpha_\eta = \lambda(\eta, \bullet)\,\mathcal{F}$:

$$\mathcal{F}(\rho_0 \oplus \rho_1)(\xi, r) = \int d\eta\; \sigma(\xi,\eta)\rho_0(\eta) \;+\; r\; \mathrm{tr}(\rho_1\, W(\xi))$$

Fourier Fourier-Weyl

Factor r absorbs dependence on arbitrary conventional phases in W.
With choice W(p,q)= W(p,0)W(0,q) get, for $\xi=(p,q)$

$$r^2\, \omega(p,q) = 1$$

$\Rightarrow$ Gelfand spectrum is a double cover of Ξ

Classical part projected out by $\frac{1}{2}\,(f(\xi,r) + f(\xi,-r))$.
Two cases for „quantum part“:

X=$\mathbf{R}^d$: global continuous square root $r_+(p,q) = \exp(i\, p.q/2)$ exists.
well defined quantum part $\frac{1}{2}(\, f(\xi, r_+(\xi)) - f(\xi, -r_+(\xi))\,)$
Gelfand spectrum has two disconnected components

X= **circle** Gelfand spectrum connected .

Basic properties:

Hausdorff & Hausdorff-Young inequalities ($||.||_p$-estimates for $*$ and $\mathcal{F}$)

(best constants a la Beckner not non-trivial)

Berezin-Lieb inequality : tr $\Phi(\rho * f) \leq$ tr $\Phi(f)$; Φ convex

basic classical$\leftrightarrow$quantum estimate for partition functions

Understand Wigner functions $\mathcal{W}\rho = \mathcal{F}_0^{-1}\mathcal{F}_1\,\rho = 2^d\,U_- * \rho$

$(\mathcal{W}\rho_1) * (\mathcal{W}\rho_2) = \rho_1 * \rho_2 \geq 0$

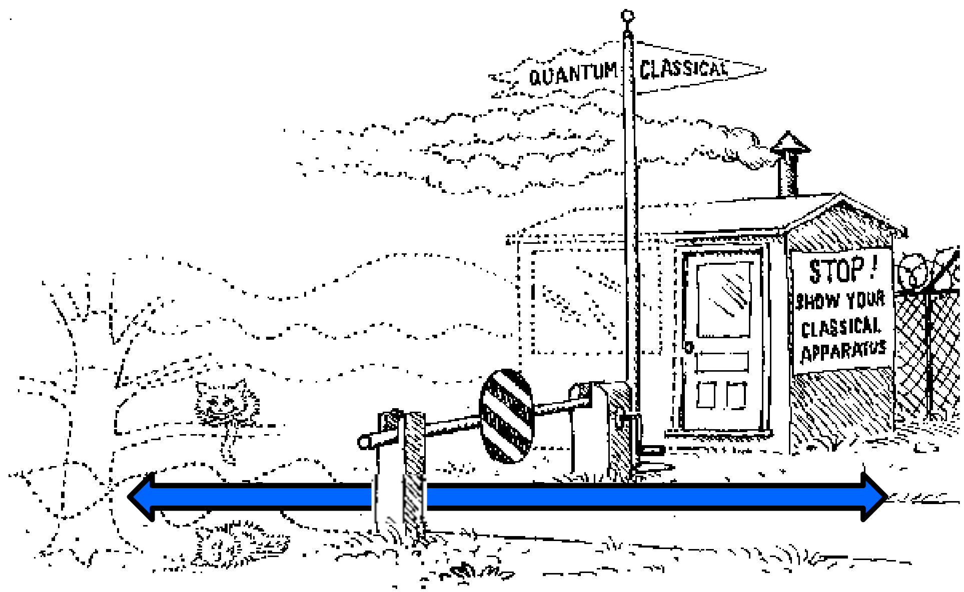
Concentration in phase space:

Lieb: Entropy($\rho_1 * \rho_2$)= min for two Gaussians of equal covariance

Unfortunately NOT: $(A_1 * A_2 * A_3)(0)$= minimal wrt rearrangements iff all sym decreasing

Correspondence
of subspaces

$\mathcal{D}_1$



$\mathcal{D}_0$

Consider translation invariant closed subspaces $\mathcal{D}$ on which translations are strongly continuous

$\rho_1 * \mathcal{D} = \rho_2 * \mathcal{D}$ if ρ_1, ρ_2 generate the same t.i. subspace of $\mathcal{T}(\mathcal{H})$ or $\mathcal{L}^1$

$\mathcal{D}_0 \subset \mathcal{L}^\infty(\Xi)$ and $\mathcal{D}_1 \subset \mathcal{B}(\mathcal{H})$ are called corresponding subspaces
if $\rho * \mathcal{D}_0 \subset \mathcal{D}_1$ and $\rho * \mathcal{D}_1 \subset \mathcal{D}_0$ for all (resp. some suitable) $\rho \in \mathcal{T}(\mathcal{H})$

ρ suitable : translates are dense in $\mathcal{T}(\mathcal{H})$.

Characterization a classical task of harmonic analysis:
(cheap import via convolution: RFW&Kiukas&Lahti&Schultz)

$\mathrm{lin}\{\alpha_\xi \rho \mid \xi \in \Xi\}$ is dense in $\mathcal{T}^1(\mathcal{H})$ iff $Z = \{\xi \mid (\mathcal{F}\rho)(\xi) = 0\}$ is empty
$\mathrm{lin}\{\alpha_\xi \rho \mid \xi \in \Xi\}$ is dense in $\mathcal{T}^2(\mathcal{H})$ iff $Z = \{\xi \mid (\mathcal{F}\rho)(\xi) = 0\}$ has measure zero
$\mathrm{lin}\{\alpha_\xi \rho \mid \xi \in \Xi\}$ is w*dense in $\mathcal{B}(\mathcal{H})$ iff $Z = \{\xi \mid (\mathcal{F}\rho)(\xi) = 0\}$ is nowhere dense

$\mathrm{lin}\{\alpha_\xi \rho \mid \xi \in \Xi\}$ dense in $\mathcal{T}^p(\mathcal{H})$ ($p \neq 1, 2, \infty$) has no characterization via Z

Efficient source of one-line characterization theorems:

$X \in \mathcal{D}_i \quad \Leftrightarrow \quad \alpha_\bullet X$ norm continuous and $\rho * X \in \mathcal{D}_{i+1}$

Easy Theorem Transfer Classical HA $\rightarrow$ QHA

$\mathcal{D}_0$	$\mathcal{D}_1$
$C_0(\Xi)$	compact ops
almost periodic	$CCR(\Xi) = C^*(\{W(\xi)\})$
lattice periodic	irrat. rotation algebra

Uniform structures on
quantum state space $\leftrightarrow$ classical state space $\leftrightarrow$ phase space
$\leftrightarrow$ compactification of Ξ

$(\mathcal{D}_0,\mathcal{D}_1)$ $\rightarrow$ functions $\langle\rho,\alpha_\bullet X\rangle$ on Ξ $\rightarrow$ $(C^*(\mathcal{D}_0), C^*(\mathcal{D}_1))$ corresponding

Fulsche's Theorem

Let $\Xi^\#$ denote the compactification of Ξ under the initial uniform structure.
Then $\alpha_\zeta X$ = extension by continuity for $\zeta\in\Xi^\#$.
Call these boundary operators.

$\partial\mathcal{D}_j := \{\alpha_\zeta X \mid X\in \mathcal{D}_j\,,\ \zeta\in\Xi^\#\backslash\Xi\}$

$\Rightarrow$ $(\partial\mathcal{D}_0, \partial\mathcal{D}_1)$ are corresponding spaces

$\mathcal{D}_0$	$\mathcal{D}_1$	$\partial\mathcal{D}_0$
$C_0(\Xi)$	compact ops	$\{0\}$
almost periodic	$C^*(\{W(\xi)\})$	same ($\Xi^\#\backslash\Xi$ dense!)
bdd unif cont	α str cont	$\Xi^\#$=Samuel comp.
		C 1, but $\mathcal{D}_0\cong$ BUC

$\alpha_\xi X$-X compact