Correspondence Theory and Toeplitz operators on Fock spaces

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Outline

- Basics of Fock spaces and their (Toeplitz) operators
- QHA in the Fock space setting
- How does the theory of Toeplitz operators benefit from QHA?
- How does QHA benefit from the theory of Toeplitz operators?

The basics of Fock spaces

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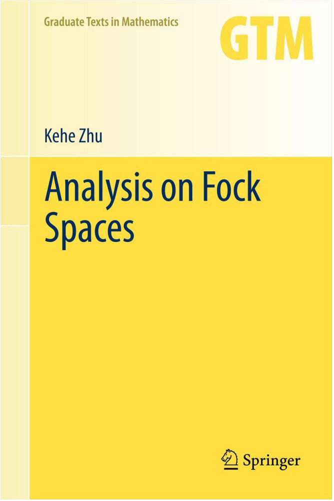
The Fock spaces are defined as $F_t^p := L_t^p \cap \mathrm{Hol}(\mathbb{C}^n)$, that is $f \in \mathrm{Hol}(\mathbb{C}^n)$ is in F_t^p iff

$$f e^{-\frac{|\cdot|^2}{2t}} \in L^p(\mathbb{C}^n).$$

Fock spaces: The standard reference

The standard reference on Fock spaces (and operator theory on them) is: Kehe Zhu: *Analysis on Fock spaces*, 2012, Springer Verlag.



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- $(F_t^p)' \cong F_t^q$ under the above inner product (with equivalent norms), where $\frac{1}{p} + \frac{1}{q} = 1$.

The standard basis

For $\alpha \in \mathbb{N}_0^n$ set

$$e_\alpha^t(z) = \frac{1}{\sqrt{\alpha! t^{|\alpha|}}} z^\alpha, \quad z \in \mathbb{C}^n.$$

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Then, $\{e_\alpha^t;\ \alpha \in \mathbb{N}_0^n\}$ is a Schauder basis for F_t^p, being orthonormal in F_t^2. In particular, $\mathcal{P}[z_1, \ldots, z_m]$ is dense in F_t^p.

The reproducing kernel structure

Let $f \in \mathrm{Hol}(\mathbb{C}^n)$ and $r > 0$. Then, Cauchy's integral formula yields

$$f(z) = \frac{1}{(\pi r^2)^n} \int_{P(z,r)} f(w)\, dw, \qquad \text{every } z \in \mathbb{C}^n.$$

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&\lesssim \|f\|_{F_t^p}.
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Explicit computations show that $K_z^t \in F_t^p$ for any p. In particular,

$$f(z) = \langle f, K_z^t \rangle_t$$

extends to any $f \in F_t^p$.

The reproducing kernel structure

A neat thing to know is the following: Every $A \in \mathcal{L}(F_t^p)$ is actually an integral operator:

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This will turn out useful later!

Intermezzo: The Bargmann transform

The Bargmann transform is the isometric isomorphism $B_t : L^2(\mathbb{R}^n) \to F_t^2$ given by

$$B_t f(z) = \left(\frac{2}{\pi t}\right)^{n/4} \int_{\mathbb{R}^n} f(x) e^{2\frac{x\cdot z}{t} - \frac{x\cdot x}{t} - \frac{z\cdot z}{t}}\ dx.$$

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The inverse can be explicitly written as

$$B_t^{-1} g(x) = \left(\frac{2}{\pi t}\right)^{n/4} \int_{\mathbb{C}^n} g(z) e^{2\frac{x \cdot \overline{z}}{t} - \frac{x \cdot x}{t} - \frac{\overline{z} \cdot \overline{z}}{2t}} \, d\mu_t(z).$$

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Theorem (Bargmann, Feichtinger, Gröchenig, Toft)

B_t^{-1} is an isometric isomorphism from F_t^p to the modulation space $M^{p,p}$.

Operators on Fock spaces

The orthogonal projection $P_t : L_t^2 \to F_t^2$ is given by

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Theorem

P_t, considered as the above integral operator, gives a bounded projection from L^p_t onto F^p_t.

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$A \mapsto \widetilde{A}$ is injective!

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Finally, we will encounter the parity operator $Uf(z) = f(-z)$.

The group actions

On the classical side, QHA in the Fock space picture works completely analogous on $L^1(\mathbb{C}^n) = L^1(\mathbb{R}^{2n})$ and $L^\infty(\mathbb{C}^n) = L^\infty(\mathbb{R}^{2n})$, respectively.

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$$\alpha_z(A) = W_z^t A W_{-z}^t.$$

Nuclear operators

An operator $A \in \mathcal{L}(F_t^p)$ is called *nuclear*, write $A \in \mathcal{N}(F_t^p)$, if there are $f_j \in F_t^p$, $g_j \in F_t^q$ such that

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$\mathcal{N}(F_t^p)$ comes with the *nuclear trace*:

$$tr(A) = \sum_{j=1}^{\infty} \langle f_j, g_j \rangle_t.$$

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need a new proof (unless $p = 2$, then the same proof can be utilized). Idea: Verify the identity by direct computations for $A = f_1 \otimes g_1$ and $B = f_2 \otimes g_2$, where $f_j, g_j \in \mathcal{P}[z_1, \ldots, z_n]$. This is done by somewhat lengthy computations involving the reproducing kernel structure.

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need a new proof (unless $p = 2$, then the same proof can be utilized). Idea: Verify the identity by direct computations for $A = f_1 \otimes g_1$ and $B = f_2 \otimes g_2$, where $f_j, g_j \in \mathcal{P}[z_1, \dots, z_n]$. This is done by somewhat lengthy computations involving the reproducing kernel structure. Then, use that finite rank operators are dense in $\mathcal{N}(F_t^p)$ and polynomials are dense in F_t^p.

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In particular, we can continue the convolution operators of QHA to one factor being in $L^\infty(\mathbb{C}^n)$ or $\mathcal{L}(F_t^p)$ by the same duality arguments.

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$$\begin{aligned}\mathcal{C}_0 &:= \{f \in L^\infty(\mathbb{C}^n);\ z \mapsto \alpha_z(f) \text{ is } \|\cdot\|_\infty\text{-cont.}\}\\ &= \mathrm{BUC}(\mathbb{C}^n)\\ \mathcal{C}_1^{p,t} &:= \{A \in \mathcal{L}(F_t^p); z \mapsto \alpha_z(A) \text{ is } \|\cdot\|_{op}\text{-cont.}\}\end{aligned}$$

Correspondence Theory

Let $\mathcal{D}_0 \subset L^\infty(\mathbb{C}^n)$ and $\mathcal{D}_1 \subset \mathcal{L}(F_t^p)$ be α-invariant. $(\mathcal{D}_0, \mathcal{D}_1)$ is said to be a pair of *Corresponding Spaces* if:

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Besides some general properties of such pairs, R. F. Werner proved that there is a 1:1 correspondence between certain spaces on the classical and operator side in the above sense. For this, recall that $A \in \mathcal{N}(F_t^p)$ is called a *regular operator* if $\mathrm{span}\{\alpha_z(A);\ z \in \mathbb{C}^n\}$ is dense in $\mathcal{N}(F_t^p)$.

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- *Let* A *be a regular operator. Then, the corresponding spaces above are given by*

$$\mathcal{D}_1 = \overline{A * \mathcal{D}_0}, \quad \mathcal{D}_0 = \overline{A * \mathcal{D}_1}$$

- *Let* A *be a regular operator,* $\mathcal{D}_0$*,* $\mathcal{D}_1$ *as above and* $f \in \mathrm{BUC}(\mathbb{C}^n)$*,* $B \in \mathcal{C}_1^{p,t}$*. Then, we have:*

$$f \in \mathcal{D}_0 \Leftrightarrow A * f \in \mathcal{D}_1, \quad B \in \mathcal{D}_1 \Leftrightarrow B * A \in \mathcal{D}_0.$$

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has dense range.

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Let us give some simple applications of QHA to the theory of Toeplitz operators!

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The initial proof by Xia only worked for $p = 2$ and filled a somewhat lengthy, very technical paper[1]

[1] J. Xia: *Localization and the Toeplitz algebra on the Bergman space*, 2015, J. Funct. Anal. 269:781-814

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Theorem (W. Bauer, J. Isralowitz '12)

Let $A \in \mathcal{L}(F_t^p)$. *Then,*

$$A \in \mathcal{K}(F_t^p) \Leftrightarrow A \in \mathcal{T}^{p,t} \text{ and } \widetilde{A} \in C_0(\mathbb{C}^n).$$

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Note that this argument involving the Correspondence Theorem is also significantly shorter than the original proof[2]

[2] W. Bauer, J. Isralowitz: *Compactness characterization of operators in the Toeplitz algebra of the Fock space F_α^p*, 2012, J. Funct. Anal. 263:1323-1355

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> *It would be particularly interesting with what qualifications (if any) "a C^*-algebra" can be added to the above list.*

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For an improvement of the theorem, we introduce the following group action of $\mathbb{R}_+$ on $L^\infty(\mathbb{C}^n)$:

$$\delta_\lambda f(z) = f(\lambda z), \quad \lambda > 0.$$

QHA and Toeplitz operators

Theorem (RF '20, S. Wu - X. Zhao '21)

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We say that a subspace $\mathcal{D}_0 \subset L^\infty(\mathbb{C}^n)$ is δ-invariant if $\delta_\lambda f \in \mathcal{D}_0$ whenever $f \in \mathcal{D}_0$, $\lambda > 0$.

Theorem (RF '21)

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QHA and Toeplitz operators

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QHA and Toeplitz operators

Let us sketch the proof of one of the statements. We want to show that if $\mathcal{D}_0$ is an α-invariant closed subalgebra of $\mathrm{BUC}(\mathbb{C}^n)$, then $\mathcal{T}_{lin}^{p,t}(\mathcal{D}_0)$ is a Banach algebra.

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We present the proof of Wu and Zhao[3] in the context of QHA.

[3] S. Wu and X. Zhao: Toeplitz algebras over Fock and Bergman spaces, arXiv:2105.03950

QHA and Toeplitz operators

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Recall that each $A \in \mathcal{L}(F_t^p)$ is an integral operator:

$$Af(z) = \int_{\mathbb{C}^n} f(w)\langle AK_w^t, K_z^t\rangle_t \, d\mu_t(w).$$

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$$Af(z) = \int_{\mathbb{C}^n} f(w)\langle AK_w^t, K_z^t\rangle_t \, d\mu_t(w).$$

There is a well-known formula on the kernel for the product of integral operators:

$$k_{AB}(w,z) = \int_{\mathbb{C}^n} k_A(w,\xi)k_B(\xi,z) \, d\mu_t(\xi).$$

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Hence, for $A, B \in \mathcal{L}(F_t^p)$, AB is given as an integral operator with integral kernel

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$$\widetilde{AB}(z) = \frac{1}{(\pi t)^n} \int_{\mathbb{C}^n} \langle Ak_z^t, k_\xi^t \rangle_t \langle Bk_\xi^t, k_z^t \rangle_t \; d\xi.$$

QHA and Toeplitz operators

Now, for $A \in \mathcal{T}^{p,t}_{lin}(\mathcal{D}_0)$ and $f \in \mathcal{D}_0$:

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QHA and Toeplitz operators

Now, for $A \in \mathcal{T}^{p,t}_{lin}(\mathcal{D}_0)$ and $f \in \mathcal{D}_0$:

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QHA and Toeplitz operators

Now, for $A \in \mathcal{T}_{lin}^{p,t}(\mathcal{D}_0)$ and $f \in \mathcal{D}_0$:

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QHA and Toeplitz operators

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Observe that

$$\langle AW_z^t k_v^t, W_z^t k_w^t\rangle_t = A * (k_{-v}^t \otimes k_{-w}^t)(z)$$

QHA and Toeplitz operators

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(and similarly for the other inner product), hence

$$\langle AW_{(\cdot)}^t 1, W_{(\cdot)}^t k_\xi^t \rangle_t, \langle T_f^t W_{(\cdot)}^t k_\xi^t, W_{(\cdot)}^t 1 \rangle_t \in \mathcal{D}_0.$$

QHA and Toeplitz operators

Since we assumed that $\mathcal{D}_0$ is an algebra,

$$\langle AW^t_{(\cdot)}1, W^t_{(\cdot)}k^t_\xi\rangle_t \cdot \langle T^t_f W^t_{(\cdot)}k^t_\xi, W^t_{(\cdot)}1\rangle_t \in \mathcal{D}_0$$

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Further, we have the estimate

$$|\langle AW_z^t 1, W_z^t k_\xi^t \rangle_t \cdot \langle T_f^t W_z^t k_\xi^t, W_z^t 1 \rangle_t| \lesssim \|A\|_{op} |\langle T_{\alpha_{-z}(f)} k_\xi^t, 1 \rangle_t|$$

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&\quad = \|A\|_{op} \|f\|_\infty e^{-\frac{|\xi|^2}{4t}}.
\end{aligned}$$

We therefore obtain

$$\widetilde{AT_f^t} = \int_{\mathbb{C}^n} \langle AW_{(\cdot)}^t 1, W_{(\cdot)}^t k_\xi^t \rangle_t \langle T_f^t W_{(\cdot)}^t k_\xi^t, W_{(\cdot)}^t 1 \rangle_t \ d\xi$$

and the right-hand side exists as a Bochner integral in $\mathcal{D}_0$.

QHA and Toeplitz operators

We therefore obtain

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and the right-hand side exists as a Bochner integral in $\mathcal{D}_0$. Hence, by the Correspondence Theorem, $AT_f^t \in \mathcal{T}_{lin}^{p,t}(\mathcal{D}_0)$. Analogously, one can show that $T_f^t A \in \mathcal{T}_{lin}^{p,t}(\mathcal{D}_0)$. Therefore, $\mathcal{T}_{lin}^{p,t}(\mathcal{D}_0)$ is an algebra.

Thank you for your attention!