Algebraic $K$-theory of group rings and topological cyclic homology

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Outline

1. Conjectures
2. Theorems
3. Proofs
This is an overview of joint work with

- Wolfgang Lück,
- Holger Reich and
- Marco Varisco.
Outline

1 Conjectures

2 Theorems

3 Proofs
Conjecture (Borel (1953))

Let $G$ be any discrete group. Any two closed manifolds of the homotopy type of $BG$ are homeomorphic.

This is an analogue of the Poincaré conjecture for aspherical manifolds.
Novikov conjecture

The integral Pontryagin classes $p_i(TM) \in H^{4i}(M; \mathbb{Z})$ are not topological invariants, but the rational Pontryagin classes are. Consider a map $u : M \to BG$ from an $n$-manifold to a classifying space. The higher $x$-signature, for $x \in H^{n-4i}(BG; \mathbb{Q})$, is the rational number

$$\text{sign}_x(M, u) = \langle L_i(TM) \cup u^*(x), [M] \rangle.$$

Conjecture (Novikov (1970))

If $h : M' \to M$ is an orientation-preserving homotopy equivalence, then $\text{sign}_x(M, u) = \text{sign}_x(M, uh)$. 
Conjecture (Novikov, reformulated)

The L-theory assembly map

\[ a^L : BG_+ \wedge \mathbb{L}(\mathbb{Z}) \longrightarrow \mathbb{L}(\mathbb{Z}[G]) \]

is rationally injective, i.e., the induced homomorphism

\[ a_*^L \otimes \mathbb{Q} : H_*(BG; L_*(\mathbb{Z})) \otimes \mathbb{Q} \longrightarrow L_*(\mathbb{Z}[G]) \otimes \mathbb{Q} \]

is injective in each degree.
Conjecture (Hsiang (1983))

If $G$ is a torsion-free group, and $BG$ has the homotopy type of a finite CW complex, then the $K$-theory assembly map

$$a^K : BG_+ \wedge K(\mathbb{Z}) \longrightarrow K(\mathbb{Z}[G])$$

is a rational equivalence.
A *family* $\mathcal{F}$ of subgroups of $G$ is a collection of subgroups, closed under conjugation with elements of $G$ and passage to subgroups.

Let $E\mathcal{F}$ denote the universal $G$-CW space with stabilizers in $\mathcal{F}$. Universality amounts to the condition that $E\mathcal{F}^H$ is contractible for each $H \in \mathcal{F}$. 

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The orbit category

**Definition**

The *orbit category* $\text{Or } G$ has as objects the homogeneous $G$-spaces $G/H$, and as morphisms the $G$-maps.

- The rule $G/H \mapsto E\mathbb{F}^H$ defines a contravariant functor $E\mathbb{F}^?_*$ from $\text{Or } G$ to spaces.
- The rule $G/H \mapsto K(\mathbb{Z}[H])$ can be extended to a covariant functor $K(\mathbb{Z}[?])$ from $\text{Or } G$ to spectra.

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The smash product

\[ E \mathcal{F}_+ \wedge_{\text{Or} G} K(\mathbb{Z}[-]) = E \mathcal{F}_+^? \wedge_{\text{Or} G} K(\mathbb{Z}[?]) \]

is a spectrum defined as a homotopy coend. The \( G \)-map \( E \mathcal{F} \to * \) induces a natural map

\[ a^K : E \mathcal{F}_+ \wedge_{\text{Or} G} K(\mathbb{Z}[-]) \to * + \wedge_{\text{Or} G} K(\mathbb{Z}[-]) \cong K(\mathbb{Z}[G]) , \]

which we call the \( K \)-theory assembly map for \( \mathcal{F} \).
A group is called *virtually cyclic* if it contains a (finite or infinite) cyclic subgroup of finite index. Let $G$ be any discrete group, and let $\mathcal{VCyc}$ be the family of virtually cyclic subgroups of $G$.

**Conjecture (Farrell-Jones (1993))**

The $K$-theory assembly map for $\mathcal{VCyc}$,

$$a^K: E_{\mathcal{VCyc}} \wedge_{\text{Or } G} K(\mathbb{Z}[-]) \to K(\mathbb{Z}[G]),$$

is an equivalence.
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The Bökstedt–Hsiang–Madsen theorem

Theorem (Bökstedt–Hsiang–Madsen (1993))

Let $G$ be a discrete group such that condition $(H')$ holds.

$(H')$ $H_\ast(BG; \mathbb{Z})$ is of finite type.

Then the connective $K$-theory assembly map

$$a^K : BG_+ \wedge K(\mathbb{Z}) \to K(\mathbb{Z}[G])$$

is rationally injective.
Theorem (Lück–Reich–Rognes–Varisco)

Let $G$ be a discrete group such that conditions (H) and (K) hold for each finite cyclic subgroup $C$ of $G$:

(H) $H_*(BZ_G C; \mathbb{Z})$ is of finite type, where $Z_G C$ is the centralizer of $C$ in $G$;

(K) The canonical map $K(\mathbb{Z}[C]) \to \prod_p K(\mathbb{Z}_p[C])^\wedge_p$ is rationally injective in each degree, where $p$ ranges over all primes.

Then the connective $K$-theory Farrell–Jones assembly map

$$a^K : E^\Lambda_{\text{Cyc}_+} \wedge_{\text{Or}_G} K(\mathbb{Z}[-]) \to K(\mathbb{Z}[G])$$

is rationally injective.
Condition (K) is known to hold when $C$ is the trivial group, which is why there is no explicit condition (K’) in the result of Bökstedt–Hsiang–Madsen.

Condition (K) holds in degrees $t \leq 1$; in degrees $t \geq 2$ it is expected to hold in all cases, and would follow from the Schneider conjecture (1979), generalizing Leopoldt’s conjecture from $K_1$ to $K_t$.

Condition (H), which encompasses Condition (H’), appears to be an intrinsic limitation of the cyclotomic trace method as applied to this problem.
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Let $\mathcal{Fin}$ be the family of finite subgroups of $G$.

**Proposition (Grunewald (2008))**

The family comparison map

$$E^{\mathcal{Fin}_+} \wedge_{Or G} K(\mathbb{Z}[-]) \longrightarrow E^{\mathcal{Vyc}_+} \wedge_{Or G} K(\mathbb{Z}[-])$$

is a rational equivalence.
Let $S$ be the sphere spectrum.

**Proposition**

*The linearization maps*

$$E_{\text{Fin}_{+}} \wedge_{\text{Or}_G} K(S[-]) \longrightarrow E_{\text{Fin}_{+}} \wedge_{\text{Or}_G} K(\mathbb{Z}[-])$$

and

$$K(S[G]) \longrightarrow K(\mathbb{Z}[G])$$

are rational equivalences.
Summary of first reductions

\[ E^\vee \mathcal{C}yc_+ \wedge_{\text{Or } G} K(\mathbb{Z}[-]) \xrightarrow{a^K} K(\mathbb{Z}[G]) \]
\[ \cong_{\mathbb{Q}} \]

\[ E \mathcal{F}in_+ \wedge_{\text{Or } G} K(\mathbb{Z}[-]) \xrightarrow{a^K} K(\mathbb{Z}[G]) \]
\[ \cong_{\mathbb{Q}} \]

\[ E \mathcal{F}in_+ \wedge_{\text{Or } G} K(\mathbb{S}[-]) \xrightarrow{a^K} K(\mathbb{S}[G]) \]
\[ \cong_{\mathbb{Q}} \]
The cyclotomic trace map to topological cyclic homology gives a natural transformation

$$\text{trc}: \mathcal{K}(\mathbb{S}[-]) \longrightarrow \mathcal{TC}(\mathbb{S}[-]; p)$$

of functors from $\text{Or } G$ to spectra.

$$E \mathcal{Fin}_+ \wedge_{\text{Or } G} \mathcal{K}(\mathbb{S}[-]) \xrightarrow{a^K} \mathcal{K}(\mathbb{S}[G])$$

$$1 \wedge \text{trc} \quad \downarrow$$

$$E \mathcal{Fin}_+ \wedge_{\text{Or } G} \mathcal{TC}(\mathbb{S}[-]; p) \xrightarrow{a^{TC}} \mathcal{TC}(\mathbb{S}[G]; p)$$

$$\text{trc} \quad \downarrow$$
The role of condition (K)

**Proposition (Hesselholt–Madsen)**

Let $C$ be a finite group. If $K(\mathbb{Z}[C]) \to K(\mathbb{Z}_p[C])^\wedge$ is rationally injective, then so is $\text{trc}: K(\mathcal{S}[C]) \to TC(\mathcal{S}[C]; p)$.

**Proposition (Lück)**

If the above holds for each finite cyclic subgroup $C$ of $G$, then

$$E\mathcal{F}in_+ \wedge_{\text{Or}_G} K(\mathcal{S}[-]) \to E\mathcal{F}in_+ \wedge_{\text{Or}_G} TC(\mathcal{S}[-]; p)$$

is also rationally injective.
In the case of the trivial family $\mathcal{F} = \{e\}$, the lower horizontal map

$$a^{TC} : BG_+ \wedge TC(\mathbb{S}; p) \rightarrow TC(\mathbb{S}[G]; p)$$

is the $TC$-assembly map considered by [BHM]. It does not split quite as claimed in Madsen’s survey (1994).
The homotopy pullback square

There is a homotopy Cartesian square

\[
\begin{array}{ccc}
TC(\mathbb{S}[G]; p) & \xrightarrow{\alpha} & C(\mathbb{S}[G]; p) \\
\downarrow{\beta} & & \downarrow{\text{trf}} \\
THH(\mathbb{S}[G]) & \xrightarrow{1 - \Delta_p} & THH(\mathbb{S}[G])
\end{array}
\]

where the Bökstedt–Hsiang–Madsen functor

\[
C(\mathbb{S}[G]; p) = \text{holim}_{n \geq 1} THH(\mathbb{S}[G])_{hC_p^n}
\]

is the homotopy limit over the transfer maps.
The composite $\beta \circ \text{trc}: K(\mathbb{S}[G]) \to \text{THH}(\mathbb{S}[G])$ is the Waldhausen trace map, in the form given by Bökstedt.

There is a natural equivalence

$$\text{THH}(\mathbb{S}[G]) \simeq \mathbb{S}[B_{\text{cy}}(G)],$$

where $B_{\text{cy}}(G)$ is the cyclic bar construction on $G$.

$\Delta_p: B_{\text{cy}}(G) \to B_{\text{cy}}(G)$ is the $p$-th power map.
A decomposition

There is a decomposition

$$B^c_y(G) = \bigsqcup_{[g]} B^c_y(g)(G)$$

where $[g]$ ranges over the conjugacy classes of elements in $G$, and $B^c_y(g)(G)$ is the path component that contains the vertex $g$.

The $p$-th power map $\Delta_p$ takes $B^c_y(g)(G)$ to $B^c_y(g^p)(G)$. 
The difficulty

The $THH$-assembly map

$$a^{THH} : BG_+ \wedge THH(S) \rightarrow THH(S[G])$$

is induced by the inclusion $BG \cong B_{et}^{cy}(G) \rightarrow B^{cy}(G)$. It is split by the evident retraction $pr : B^{cy}(G)_+ \rightarrow BG_+$, but

$$pr : THH(S[G]) \rightarrow BG_+ \wedge THH(S)$$

is not in general compatible with the $p$-th power map $\Delta_p$. This does not produce a map

$$pr : TC(S[G]; p) \rightarrow BG_+ \wedge TC(S; p)$$

splitting the $TC$-assembly map.
The original, correct, strategy of Bökstedt–Hsiang–Madsen, does not split the assembly map $a^{TC}$ but the assembly map

$$a^C : BG_+ \wedge C(\mathbb{S}; p) \longrightarrow C(\mathbb{S}[G]; p)$$

for the functor $C$. Hence we must construct a natural transformation

$$\alpha : TC(\mathbb{S}[-]; p) \longrightarrow C(\mathbb{S}[-]; p)$$

of functors from $\text{Or } G$ to spectra. This requires natural Segal–tom Dieck splittings and Adams transfer equivalences, constructed by Reich–Varisco (2014).
Reduction to $C$ and $THH$

$$E \mathcal{F}in_+ \wedge_{\text{Or } G} TC(\mathbb{S}[-]; p) \xrightarrow{a^{TC}} TC(\mathbb{S}[G]; p)$$

$$E \mathcal{F}in_+ \wedge_{\text{Or } G} (C(\mathbb{S}[-]; p) \vee T(\mathbb{S}[-])) \xrightarrow{a^{CV T}} C(\mathbb{S}[G]; p) \vee T(\mathbb{S}[G])$$

(We sometimes abbreviate $THH$ to $T$ on this page, and the next.)

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Rational injectivity of $\alpha \lor \beta$

Proposition ([BHM])

Let $D$ be a finite group. The map

$$\alpha \lor \beta : TC(S[D]; p) \longrightarrow C(S[D]; p) \lor THH(S[D])$$

is rationally injective in non-negative degrees.

Proposition (Lück)

The map

$$E\mathcal{F}in_+ \wedge_{Or G} TC(S[-]; p) \longrightarrow E\mathcal{F}in_+ \wedge_{Or G} (C(S[-]; p) \lor T(S[-]))$$

is rationally injective in non-negative degrees.
The $\mathcal{F}$-part of $THH$

For any family $\mathcal{F}$ let

$$B^\text{cy}_\mathcal{F}(G) = \bigsqcup_{\langle g \rangle \in \mathcal{F}} B^\text{cy}_{\langle g \rangle}(G)$$

be the union of the path components in $B^\text{cy}(G)$ that contain the vertices $(g)$ such that the cyclic group $\langle g \rangle$ is a member of the family $\mathcal{F}$.

The $\mathcal{F}$-part of $THH(\mathbb{S}[G])$ satisfies

$$THH_\mathcal{F}(\mathbb{S}[G]) \simeq \mathbb{S}[B^\text{cy}_\mathcal{F}(G)].$$
Splitting the $THH$-assembly map for $\mathcal{F}$

The inclusion $B^{cy}_\mathcal{F}(G) \to B^{cy}(G)$, and the projection $\text{pr}_\mathcal{F} : B^{cy}(G)_+ \to B^{cy}_\mathcal{F}(G)_+$ make the $\mathcal{F}$-part a retract of $THH(S[-])$.

**Proposition**

The left hand vertical map and the lower horizontal map in the commutative square

\[
\begin{array}{ccc}
E\mathcal{F}_+ \wedge \text{Or } G & THH(S[-]) & THH(S[G]) \\
\downarrow 1 \wedge \text{pr}_\mathcal{F} & \sim & \sim \\
E\mathcal{F}_+ \wedge \text{Or } G & THH_\mathcal{F}(S[-]) & THH_\mathcal{F}(S[G])
\end{array}
\]

are stable equivalences.
Splitting the $C$-assembly map for $\mathcal{F}$

\[
\begin{align*}
E \mathcal{F}_+ \wedge_{\text{Or } G} C(S[-]; p) & \xrightarrow{a^C} C(S[\mathbb{G}]; p) \\
\text{holim}_n (E \mathcal{F}_+ \wedge_{\text{Or } G} \text{THH}(S[-])_{h\mathbb{C}_p}) & \xrightarrow{\sim} C(S[\mathbb{G}]; p) \\
\text{holim}_n (E \mathcal{F}_+ \wedge_{\text{Or } G} \text{THH}_{\mathcal{F}}(S[-])_{h\mathbb{C}_p}) & \xrightarrow{\sim} C_{\mathcal{F}}(S[\mathbb{G}]; p)
\end{align*}
\]
Proposition (Lück–Reich–Varisco (2003))

Assuming condition (H),

\[ \kappa : E \mathcal{F}^+ \wedge_{\text{Or } G} C(\mathbb{S}[-]; p) \rightarrow \text{holim}_n (E \mathcal{F}^+ \wedge_{\text{Or } G} \text{THH}(\mathbb{S}[-])_{hC_{p^n}}) \]

is an equivalence for \( \mathcal{F} \) the family of finite cyclic subgroups of \( G \), hence also for the family \( \mathcal{F}\text{in} \) of finite subgroups of \( G \).

This implies that \( a^C \) for \( \mathcal{F}\text{in} \) is split injective. Q.E.D.
Summary of all reductions ($T = \text{THH}$)

\[
E \vee \text{Cyc}_+ \wedge_{\text{Or } G} K(\mathbb{Z}[-]) \xrightarrow{a^K} K(\mathbb{Z}[G]) \\
\cong \mathbb{Q} \\
E \text{Fin}_+ \wedge_{\text{Or } G} K(\mathbb{Z}[-]) \xrightarrow{a^K} K(\mathbb{Z}[G]) \\
\cong \mathbb{Q} \\
E \text{Fin}_+ \wedge_{\text{Or } G} K(\mathbb{S}[-]) \xrightarrow{a^K} K(\mathbb{S}[G]) \\
\cong \mathbb{Q} \\
E \text{Fin}_+ \wedge_{\text{Or } G} TC(\mathbb{S}[-]; p) \xrightarrow{a^{TC}} TC(\mathbb{S}[G]; p) \\
\text{Q-inj.} \\
\text{trc} \\
E \text{Fin}_+ \wedge_{\text{Or } G} (C(\mathbb{S}[-]; p) \lor T(\mathbb{S}[-])) \xrightarrow{a^{C \lor T}} C(\mathbb{S}[G]; p) \lor T(\mathbb{S}[G]) \\
\text{non-neg. Q-inj.} \\
\alpha \lor \beta
\]