Abel–Jacobi maps and homotopy theory

Nordic Topology Meeting 2014

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joint work with Michael J. Hopkins
Parametrizing the circle:

Consider the unit circle $S: x^2 + y^2 = 1$ with $O=(1,0)$. 

\[ \text{Diagram showing a circle with center } O \text{ and a point } P. \]
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We can parametrize points on $S$ by the arclength $a$. 
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We can parametrize points on \( S \) by the arclength \( a \).

To calculate \( a \) we need to evaluate the integral

\[
I(y) := \int_0^y \frac{1}{\sqrt{1-t^2}} \, dt.
\]
From Weierstrass to Riemann:
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Taking the square root $\sqrt{(1-y^2)}$ is not a single-valued function.
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Weierstrass: Multi-valued functions.
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Weierstrass: Multi-valued functions.

Riemann: We should use a different type of domain, Riemann surfaces.
The complex circle:
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We should consider \[ S(C) = \{(x,y) \in C^2 \mid x^2 + y^2 = 1\} \]

with differential \( \omega = dy/x \).
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The integral becomes \( I(P) = \int_{0}^{p} \omega. \)
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The integral becomes $I(P) = \int_0^P \omega$.

We obtain a well-defined holomorphic and bijective function

$$S(C) \rightarrow C/2\pi\mathbb{Z}$$

$$P \mapsto \int_0^P \omega \mod 2\pi\mathbb{Z}.$$
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2\(\pi\) is the period of the circle.
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We would like to evaluate the integral

$$I(t):= \int_0^t \frac{1}{\sqrt{f(x)}} \, dx.$$
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Set $y^2 = f(x)$ and consider the space $E(C)$ of complex solutions. Set $\omega = \frac{dx}{y}$. 
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Euler's addition formula:
\[
\int_0^P \omega + \int_0^Q \omega = \int_0^{P+Q} \omega
\]
where \( P+Q \) refers to the group structure on the "elliptic curve" \( y^2 = f(x) \).
Calculating $I(P) := \int_{0}^{P} \omega \ "on \ y^2 = f(x)\ "$:
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Calculating the integral depends on the choice of a homotopy class of paths from \( 0 \) to \( P \).
Calculating $I(P) := \int_0^P \omega$ “on $y^2 = f(x)$”:

Calculating the integral depends on the choice of a homotopy class of paths from 0 to $P$.

Hence $P \mapsto I(P)$ is really a function on the universal cover $E(C)$ of $E(C)$:

$$
\begin{array}{ccc}
E(C) & \rightarrow & C \\
\downarrow & & \downarrow \\
E(C) & \rightarrow & C \\
\end{array}
$$

Euler: this is a group homomorphism.
The Jacobian of $E(C)$ and the Abel-Jacobi map:
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Choose two closed loops $\gamma_1$ and $\gamma_2$ which form a basis of $H_1(E(C);\mathbb{Z}) \approx H_1(S^1 \times S^1;\mathbb{Z}) \approx \mathbb{Z} \times \mathbb{Z}$. 
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Let $\omega_1 = \int_{\gamma_1} \omega$ and $\omega_2 = \int_{\gamma_2} \omega$ be the periods of $\omega$. 

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Let $\omega_1 = \int_{\gamma_1} \omega$ and $\omega_2 = \int_{\gamma_2} \omega$ be the periods of $\omega$.

The map $P \mapsto \int_0^P \omega$ defines an isomorphism

$$E(C) \rightarrow \mathbb{C}/(\mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2) \approx \text{Jac}(E).$$
The Abel–Jacobi Theorem:
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Picking a basepoint $P_0 \in S$ yields a map

$$
\mu: P \mapsto \left( \int_{P_0}^P \omega_1, \ldots, \int_{P_0}^P \omega_g \right)
$$
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Abel–Jacobi Theorem: The induced map

$$\mu: \text{Div}^0(S)/\sim \to \mathbb{C}^g/\Lambda \approx \text{Jac}(S)$$

is an isomorphism.
Griffiths: Use Hodge theory in higher dimensions
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Then $\left( \omega \mapsto \int_{\Gamma} \omega \right) \in F^{n-p+1}H^{2n-2p+1}(X;\mathbb{C})^*$. 
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Then

$$\left( \omega \mapsto \int_{\Gamma} \omega \right) \in F^{n-p+1}H^{2n-2p+1}(X;\mathbb{C})^*.$$

But the value depends on the choice of $\Gamma$. 
The intermediate Jacobian of Griffiths and the Abel-Jacobi map:
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We obtain a well-defined map

\[ \mu : Z^p_h(X) \rightarrow J^{2p-1}(X) = F^{n-p+1}H^{2n-2p+1}(X;\mathbb{C})^*/H_{2n-2p+1}(X;\mathbb{Z}) \]

\[ \approx H^{2p-1}(X;\mathbb{C})/(F^pH^{2p-1}(X)+H^{2p-1}(X;\mathbb{Z})) \]
The intermediate Jacobian of Griffiths and the Abel-Jacobi map:

We obtain a well-defined map

\[ Z \rightarrow \int_{\Gamma} \Gamma \] for some \( \Gamma \) with \( Z = \partial \Gamma \)

\[ \mu : Z^p_h(X) \rightarrow J^{2p-1}(X) = F^{n-p+1}H^{2n-2p+1}(X;C)/H_{2n-2p+1}(X;Z) \]

\[ \cong H^{2p-1}(X;C)/(FpH^{2p-1}(X)+H^{2p-1}(X;Z)) \]

\( J^{2p-1}(X) \) is a complex torus and is called Griffiths' intermediate Jacobian.
Hodge classes and Deligne cohomology:
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\[ \text{CH}^p(X) \ni Z \subseteq X \]

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\[ [Z_{sm}]_{\text{fund}} \]
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Kernel of $\text{cl}_H \subset CH^p(X)$ \quad $Z \subset X$

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$\mathcal{J}^{2p-1}(X) \to H_D^{2p}(X; \mathbb{Z}(p)) \to \text{Hdg}^{2p}(X)$
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$0 \to J^{2p-1}(X) \to H_D^{2p}(X;\mathbb{Z}(p)) \to \text{Hdg}^{2p}(X) \to 0$

“Deligne cohomology sees everything.”
Some more motivation: Totaro's factorization
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\[ \text{CH}^p(X) \xrightarrow{\text{cl}_H} \text{H}^{2p}(X;\mathbb{Z}) \]

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\[ \xrightarrow{\text{cl}_{\text{MU}}} \]

\[ \text{MU}^{2p}(X) \]

\[ \xrightarrow{\varnothing} [\mathbb{Z}_{\text{sm}}]_{\text{H-fund. class}} \]

\[ \xrightarrow{\text{cl}_{\text{MU}}} [\mathbb{Z}_{\text{sm}}]_{\text{MU-fund. class}} \]
Some more motivation: Totaro's factorization

\[ \text{CH}^p(X) \xrightarrow{\text{cl}_H} H^{2p}(X;\mathbb{Z}) \]

\[ \mathbb{Z} \subset X \xrightarrow{\text{cl}_{\text{MU}}} \text{MU}^{2p}(X) \otimes_{\text{MU}^*} \mathbb{Z} \]

\[ \{Z_{\text{sm}}\}_{\text{H-fund. class}} \]
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\[ [\mathbb{Z}_{\text{sm}}]_{\text{MU-fund. class}} \]

\[ [\mathbb{Z}_{\text{sm}}]_{H\text{-fund. class}} \]

This is very useful!
Consequences:

\[ \text{CH}^p(X) \xrightarrow{\text{cl}_H} H^{2p}(X;\mathbb{Z}) \]

\[ \text{cl}_{\text{MU}} \quad \text{MU}^{2p}(X) \otimes_{\text{MU}^*} \mathbb{Z} \]
Consequences:

- A topological obstruction on the image of \( \text{cl}_H \): the image of \( \text{cl}_H \) is contained in the image of \( \vartheta \).
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- We can study the kernel of $\text{cl}_H$.

Totaro’s strategy: find elements in the kernel of $\vartheta$ that are in the image of $\text{cl}_H$. 

\[
\begin{array}{c}
\text{CH}^p(X) \\ \downarrow \text{cl}_H \\
H^{2p}(X; \mathbb{Z}) \\
\downarrow \vartheta \\
MU^{2p}(X) \otimes_{MU^*} \mathbb{Z}
\end{array}
\]
Consequences:

- A topological obstruction on the image of $\text{cl}_H$: the image of $\text{cl}_H$ is contained in the image of $\vartheta$.
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Totaro's strategy: find elements in the kernel of $\vartheta$ that are in the image of $\text{cl}_{MU}$.

\[
\begin{align*}
\text{CH}^p(X) & \xrightarrow{\text{cl}_H} H^{2p}(X;\mathbb{Z}) \\
\text{cl}_{MU} & \downarrow \\
\text{MU}^{2p}(X) \otimes_{MU^*} \mathbb{Z} & \xrightarrow{\vartheta} \\
\end{align*}
\]
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\[
0 \rightarrow J^{2p-1}(X) \rightarrow H_D^{2p}(X; \mathbb{Z}(p)) \rightarrow \text{Hdg}^{2p}(X) \rightarrow 0
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Deligne cohomology:
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a homotopy cartesian square of sheaves of complexes.
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\mathbb{Z} \quad \text{a homotopy cartesian square of sheaves of complexes.}
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\[
\begin{array}{c}
\mathbb{Z} \\
\Omega^*_\text{hol}
\end{array}
\]

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Deligne cohomology: Given an integer \( p \geq 0 \).

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\[
\begin{array}{ccc}
\mathbb{Z} & \xrightarrow{	ext{a homotopy cartesian square}} & \text{of sheaves of complexes.} \\
\Omega_{\text{hol}}^* & \downarrow & \\
\end{array}
\]
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\[
\begin{array}{ccc}
\Omega^*_{\text{hol}} & \longrightarrow & \mathbb{Z} \\
\downarrow & & \downarrow \\
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$$\Omega^{\bullet \geq p}_{\text{hol}} \xrightarrow{Z} \Omega^\bullet_{\text{hol}}$$

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\[
\begin{array}{ccc}
Z_{D}(p) & \to & Z \\
\downarrow & & \downarrow \\
\Omega_{\text{hol}}^{\geq p} & \to & \Omega_{\text{hol}}^{\ast}
\end{array}
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Z_D(p) & \rightarrow & Z \\
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\Omega^{* \geq p}_{hol} & \rightarrow & \Omega^{*}_{hol}
\end{array}
\]

a homotopy cartesian square of sheaves of complexes.

Deligne cohomology is the hypercohomology of this complex, i.e.,
\[
H^n_D(X; Z(p)) = H^n(X; Z_D(p)).
\]
The construction: A homotopy cartesian square of sheaves of complexes

$$Z_D(p) \rightarrow Z$$

$$\Omega_{\text{hol}}^{* \geq p} \rightarrow \Omega_{\text{hol}}^*$$
The construction: A homotopy cartesian square of presheaves of complexes.
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\[ \Omega_{\text{hol}}^* \geq p \]

\[ Z_{D(p)} \rightarrow \rightarrow Z \]

\[ \Omega_{\text{hol}}^* \rightarrow \rightarrow \Omega_{\text{hol}}^* \]
The construction: A homotopy cartesian square of presheaves of spectra on the site of complex manifolds.
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\[ \Omega_{hol}^{* \geq p} \rightarrow \Omega_{hol}^* \]

\[ Z \rightarrow H = \text{Eilenberg-MacLane spectrum functor for complexes} \]
The construction: A homotopy cartesian square of presheaves of spectra on the site of complex manifolds

\[ \Omega_{\text{hol}}^* \geq p \quad \rightarrow \quad \Omega_{\text{hol}}^* \]

\[ HZ \quad \downarrow \quad H^* \equiv \text{Eilenberg-MacLane spectrum functor for complexes} \]
The construction: A homotopy cartesian square of presheaves of spectra on the site of complex manifolds.

\[ \Omega_{\text{hol}}^{* \geq p} \rightarrow H \Omega_{\text{hol}}^{*} \]

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\[
\begin{array}{ccc}
HZ_D(p) & \longrightarrow & HZ \\
\downarrow & & \downarrow \\
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The construction: A homotopy cartesian square of presheaves of spectra on the site of complex manifolds

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HZ_D(p) & \longrightarrow & HZ \\
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H\Omega_{\text{hol}}^{\geq p} & \longrightarrow & H\Omega_{\text{hol}}^* \\
\end{array} \]

\( H = \text{Eilenberg-MacLane spectrum functor for complexes} \)

\( HZ_D(p) \) represents Deligne cohomology in the homotopy category of presheaves of spectra.
The construction: A homotopy cartesian square of presheaves of spectra on the site of complex manifolds

\[ \begin{array}{ccc}
  HZ_D(p) & \longrightarrow & H \\
  \downarrow & & \downarrow \\
  H \Omega_{hol}^{* \geq p} & \longrightarrow & H \Omega_{hol}^{*}
\end{array} \]

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\( HZ_D(p) \) represents Deligne cohomology in the homotopy category of presheaves of spectra.
The construction: A homotopy cartesian square of presheaves of spectra on the site of complex manifolds

![Diagram](image)

$HZ_{D}(p)$ represents Deligne cohomology in the homotopy category of presheaves of spectra.

$H=\text{Eilenberg-MacLane spectrum functor for complexes}$
The construction: A homotopy cartesian square of presheaves of spectra on the site of complex manifolds

\[
\begin{align*}
\text{HZ}_D(p) & \quad \rightarrow \quad \text{MU} \\
\downarrow & \quad \downarrow \\
H\Omega_{\text{hol}}^{* \geq p} & \quad \rightarrow \quad H\Omega_{\text{hol}}^* 
\end{align*}
\]

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\begin{array}{ccc}
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H \Omega_{\text{hol}}^{* \geq p} & \rightarrow & H \Omega_{\text{hol}}^{*}(MU_C^{2*})
\end{array}
\]

\(HZ_D(p)\) represents Deligne cohomology in the homotopy category of presheaves of spectra.
The construction: A homotopy cartesian square of presheaves of spectra on the site of complex manifolds

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\[ \text{HZ}_D(p) \rightarrow \text{MU} \]

\[ H=\text{Eilenberg-MacLane spectrum functor for complexes} \]

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\[ \text{HZ}_D(p) \to \text{MU} \]

\[ \downarrow \quad \downarrow \]

\[ \to H\Omega_{\text{hol}}^*(\text{MC}^{2*}) \]

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\downarrow \\
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\downarrow \\
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Hodge filtered complex bordism:

$\text{MU}_D(p) \to \text{MU}

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\( X \) a complex manifold, and \( n, p \) integers
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\( X \) a complex manifold, and \( n, p \) integers

We define:

\[ \text{MU}_D^n(p)(X) := \text{Hom}_{\text{HoPre}}(\Sigma^\infty(X_+), \Sigma^n\text{MU}_D(p)) \]
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“HFC bordism” groups sit in long exact sequences.
A diagram of short exact sequences
(a compact complex Kähler manifold):
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(X a compact complex Kähler manifold):

\[ 0 \rightarrow J^{2p-1}(X) \rightarrow H_{D}^{2p}(X;\mathbb{Z}(p)) \rightarrow Hdg^{2p}(X) \rightarrow 0 \]
A diagram of short exact sequences
(X a compact complex Kähler manifold):

\[ 0 \rightarrow J_{\text{MU}}^{2p-1}(X) \rightarrow \text{MU}_D^{2p}(p)(X) \rightarrow \text{Hdg}_{\text{MU}}^{2p}(X) \rightarrow 0 \]

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A diagram of short exact sequences
(X a compact complex Kähler manifold):

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0 \to J^{2p-1}_{MU}(X) \to MU^{2p}(p)(X) \to Hdg^{2p}_{MU}(X) \to 0
\]

\[
0 \to J^{2p-1}(X) \to H^{2p}_D(X; \mathbb{Z}(p)) \to Hdg^{2p}(X) \to 0
\]
A diagram of short exact sequences (\(X\) a compact complex Kähler manifold):

\[
\begin{align*}
0 & \to J^{2p-1}_{M\mathbb{U}}(X) \to M\mathbb{U}^{2p}_D(p)(X) \to Hdg^{2p}_{M\mathbb{U}}(X) \to 0 \\
0 & \to J^{2p-1}(X) \to H^{2p}_D(X;\mathbb{Z}(p)) \to Hdg^{2p}(X) \to 0
\end{align*}
\]

\(J^{2p-1}_{M\mathbb{U}}(X)\) is a complex torus which we think of as a “generalized Jacobian”.
A diagram of short exact sequences
(X a compact complex Kähler manifold):

\[ 0 \rightarrow \mathcal{J}_{\text{MU}}^{2p-1}(X) \rightarrow \text{MU}_D^{2p}(p)(X) \rightarrow \text{Hdg}_{\text{MU}}^{2p}(X) \rightarrow 0 \]

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\(\mathcal{J}_{\text{MU}}^{2p-1}(X)\) is a complex torus which we think of as a
“generalized Jacobian”.

As a real Lie group it is \(\cong \text{MU}^{2p-1}(X) \otimes \mathbb{R}/\mathbb{Z}\).
There is an improved version for complex algebraic varieties:
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- Work on the Nisnevich site on $\text{Sm}_C$
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- It remedies the defects of non-compactness.
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- Transfers: a projective morphism induces a push-forward homomorphism.
Interesting maps for smooth complex varieties:
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Let $X$ be a smooth projective complex variety.
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Let $X$ be a smooth projective complex variety.

\[
\Omega^p(X) 
\xrightarrow{\Phi} \text{Hdg}_{MU}^{2p}(X) \rightarrow 0 
\]

\[
0 \rightarrow J_{MU}^{2p-1}(X) \rightarrow MU_D^{2p}(p)(X) \rightarrow \text{Hdg}_{MU}^{2p}(X) \rightarrow 0 
\]
A new Abel–Jacobi map:
Let $X$ be a smooth projective complex variety.

\[
\begin{align*}
\Omega^p(X) & \to [Y \to X] \\
& \downarrow \Phi \\
0 & \to J_{MU}^{2p-1}(X) \to MU_D^{2p}(p)(X) \to \text{Hdg}_{MU}^{2p}(X) \to 0
\end{align*}
\]
A new Abel–Jacobi map:

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$$
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$$

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Then \(W\) defines a “current” via integrals over chains in
\[
\bigoplus_{j \geq 0} \bigwedge^{n-p-j+1} H^{2n-2p-2j+1}(X;C)^* \otimes \pi_{2j} MU
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This depends on the choice of \(W\), but it becomes well-defined modulo \(\mathbf{MU}_{2n-2p+1}(X)\):
An Abel–Jacobi map: \( n = \dim X \)

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\]

This depends on the choice of \(W\), but it becomes well-defined modulo \(MU_{2n-2p+1}(X)\):

\[
\ker(\Phi) \to F^{n-p+*+1}H^{2n-2p+2*+1}(X)^* \otimes \pi_{2*}MU / MU_{2n-2p+1}(X)
\]

\[
\approx J^{2p-1}_{MU}(X)
\]
Examples:

We check that HFC bordism is able to detect interesting algebraic cobordism classes:

\[ \Omega^p(X) \]

\[ \Phi_D \]

\[ \Phi \]

\[ \text{CH}^p(X) \]

\[ \text{MU}^{2p}_D(p)(X) \]

\[ \text{MU}^{2p}(X) \]
Examples:

We check that HFC bordism is able to detect interesting algebraic cobordism classes:

\[ \exists \alpha \in \Omega^p(X) \]

\[ \begin{align*}
\text{CH}^p(X) & \quad \Phi_D \\
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\text{MU}^{2p}(X) &
\end{align*} \]
Examples:

We check that HFC bordism is able to detect interesting algebraic cobordism classes:

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Diagram:

- $0$  \(\xrightarrow{\Phi_D}\) \(\xrightarrow{\Phi}\) $\text{MU}^{2p}(X)$
- $\text{CH}^p(X)$
- $\text{MU}_{D}^{2p}(p)(X)$

Where $\Phi_D$ and $\Phi$ are the appropriate maps.
Examples:

We check that HFC bordism is able to detect interesting algebraic cobordism classes:

$$\exists \alpha \in \Omega^p(X)$$

\[
\begin{array}{ccc}
0 & \overset{\Phi_D}{\rightarrow} & \text{MU}_{D}^{2p}(p)(X) \\
\text{CH}^p(X) & \rightarrow & \Phi \\
\end{array}
\]

$$\rightarrow 0$$

\[
\begin{array}{ccc}
\Phi & \rightarrow & 0 \\
\text{MU}^{2p}(X) & \rightarrow & 0 \\
\end{array}
\]
Examples:

We check that HFC bordism is able to detect interesting algebraic cobordism classes:

\[
\exists \alpha \in \Omega^p(X)
\]

\[
\begin{array}{ccc}
0 & \not\equiv & 0 \\
CH^p(X) & \Phi_D & \Phi \\
\neq & MU_D^{2p}(p)(X) & MU^{2p}(X)
\end{array}
\]
Examples:

We check that HFC bordism is able to detect interesting algebraic cobordism classes:

\[ \exists \alpha \in \Omega^p(X) \]

where \( \Phi_D(\alpha) = 0 \) means that \( \alpha \) is detected by the new Jacobian \( J_{MU}^{2p-1}(X) \).
Wilson splittings for Hodge filtered $\text{BP}$-theory:
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We can also construct Hodge-filtered spaces corresponding to any CW-complex.
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Let $X$ be a complex manifold and $k$ an integer with $k \leq 2(p^n + ... + p + 1)$. 
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Examples: $BP_k(q)$ and $BP\langle n \rangle_k(q)$ for a prime $p$.

Let $X$ be a complex manifold and $k$ an integer with $k \leq 2(p^n + \ldots + p + 1)$.

Then the canonical map $BP^k(q)(X) \to BP\langle n \rangle^k(q)(X)$ is surjective.
Wilson splittings for Hodge filtered $BP$-theory:
We can also construct Hodge-filtered spaces corresponding to any CW-complex.

Examples: $BP_k(q)$ and $BP^{\langle n \rangle}_k(q)$ for a prime $p$.

Let $X$ be a complex manifold and $k$ an integer with $k \leq 2(p^n + \ldots + p + 1)$.

Then the canonical map $BP^k(q)(X) \to BP^{\langle n \rangle}^k(q)(X)$ is surjective.

This is very useful for analyzing the kernel of the canonical map $BP^k(q)(X) \otimes_{BP^*Z} \to H^{\langle n \rangle}_D^k(X;Z(q))$. 
Thank you!