Abel-Jacobi maps and homotopy theory

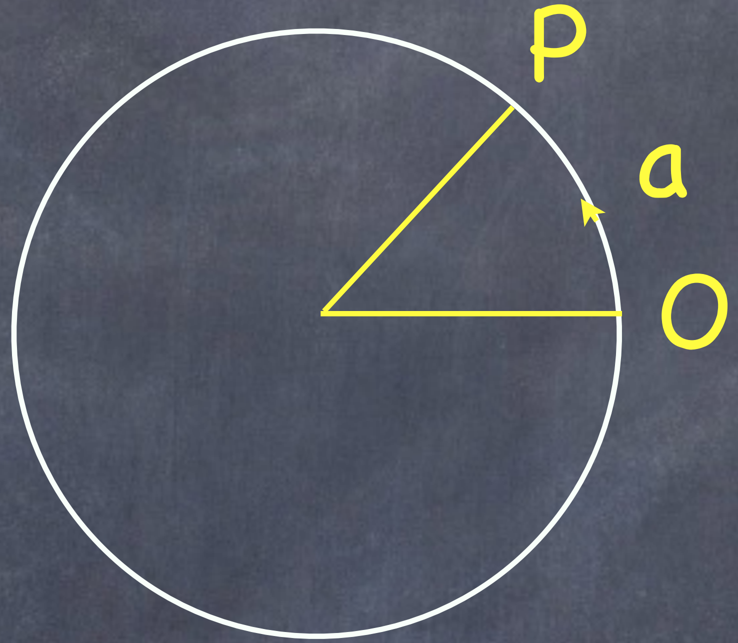
Nordic Topology Meeting 2014

Gereon Quick

joint work with Michael J. Hopkins

Parametrizing the circle:

Consider the unit circle $S: x^2 + y^2 = 1$ with $O=(1,0)$.



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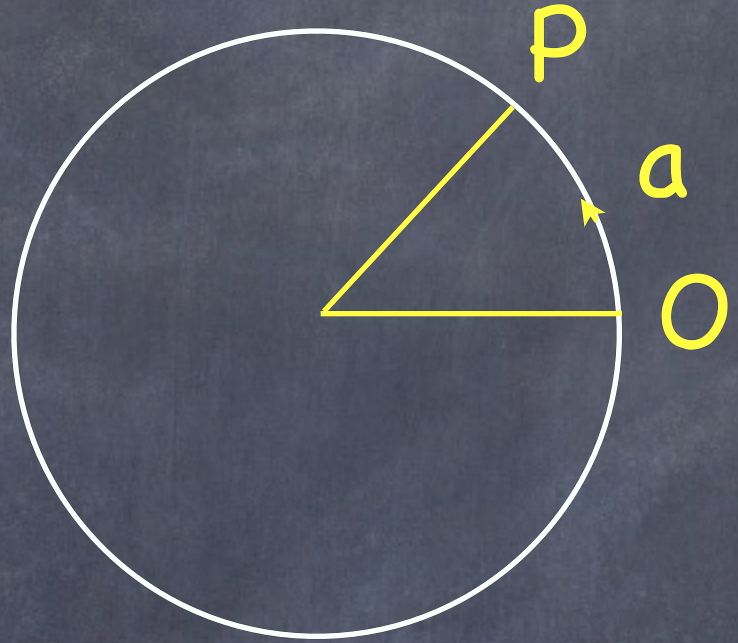

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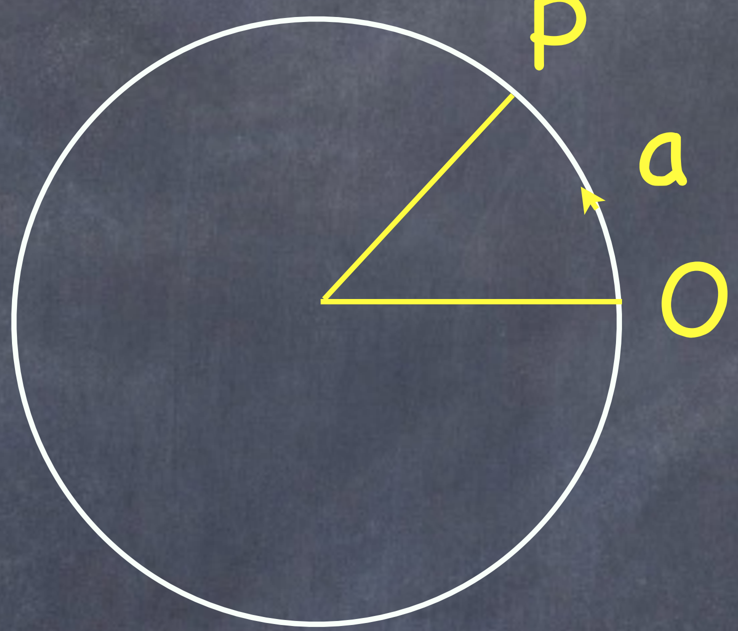

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To calculate a we need to evaluate the integral

$$I(y) := \int_0^y 1/\sqrt{(1-t^2)}\,dt.$$

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Riemann: We should use a different type of domain, Riemann surfaces.

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We obtain a well-defined holomorphic and bijective function

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2π is the period of the circle.

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Euler's addition formula:

$$\int_0^P \omega + \int_0^Q \omega = \int_0^{P+Q} \omega$$

where $P+Q$ refers to the group structure on the "elliptic curve" $y^2 = f(x)$.

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Hence $P \mapsto I(P)$ is really a function on the universal cover $\widetilde{E(C)}$ of $E(C)$:

$$\begin{tikzcd}
\widetilde{E(C)} \arrow[r] \arrow[d] & \mathbb{C} \\
E(C) &
\end{tikzcd}$$

Euler: this is a group homomorphism.

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The map $P \mapsto \int_0^P \omega$ defines an isomorphism

$$E(C) \to C/(Z\omega_1 \oplus Z\omega_2) \approx Jac(E).$$

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Abel-Jacobi Theorem: The induced map

$\mu: Div^0(S)/\sim \;\to C^g/\Lambda \approx Jac(S)$ is an isomorphism.

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Then $$\left(\omega \mapsto \int_\Gamma \omega\right) \in F^{n-p+1}H^{2n-2p+1}(X;C)^*.$$

But the value depends on the choice of Γ.

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We obtain a well-defined map

$$Z \mapsto \int_\Gamma \quad \text{for some } \Gamma \text{ with } Z=\partial\Gamma$$

$$\mu : Z^p_h(X) \to J^{2p-1}(X) = F^{n-p+1}H^{2n-2p+1}(X;C)^*/H_{2n-2p+1}(X;Z)$$

$$\approx H^{2p-1}(X;C)/(F^pH^{2p-1}(X)+H^{2p-1}(X;Z))$$

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$J^{2p-1}(X)$ is a complex torus and is called Griffiths' intermediate Jacobian.

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Kernel of $cl_H \subset CH^p(X) \xrightarrow{cl_H} Hdg^{2p}(X)$ $\quad Z \subset X \mapsto [Z_{sm}]_{fund}$

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"Deligne cohomology sees everything."

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$$Z\subset X \longmapsto [Z_{sm}]_{MU\text{-fund. class}} \longmapsto [Z_{sm}]_{H\text{-fund. class}}$$

This is very useful!

Consequences:

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a homotopy cartesian square of sheaves of complexes.

Deligne cohomology is the hypercohomology of this complex, i.e., $H_D{}^n(X;\mathbb{Z}(p)) = H^n(X;\mathbb{Z}_D(p))$.

The construction: A homotopy cartesian square of sheaves of complexes

$$\begin{tikzcd} Z_D(p) \arrow[r] \arrow[d] & Z \arrow[d] \\ \Omega_{hol}{}^{*\geq p} \arrow[r] & \Omega_{hol}{}^{*} \end{tikzcd}$$

The construction: A homotopy cartesian square of presheaves of complexes

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The construction: A homotopy cartesian square of presheaves of spectra on the site of complex manifolds

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$$\begin{array}{ccc} & & Z \\ & & \downarrow \\ \Omega_{hol}{}^{*\geq p} & \longrightarrow & \Omega_{hol}{}^{*} \end{array}$$

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"HFC bordism" groups sit in long exact sequences.

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As a real Lie group it is $\approx MU^{2p-1}(X)\otimes R/Z$.

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$$\approx J^{2p-1}_{MU}(X)$$

Examples:

We check that HFC bordism is able to detect interesting algebraic cobordism classes:

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where $\Phi_D(\alpha)=0$ means that α is detected by the new Jacobian $J_{MU}^{2p-1}(X)$.

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This is very useful for analyzing the kernel of the canonical map $BP^k(q)(X) \otimes_{BP^*} Z \to H_D{}^k(X;Z(q))$.

Thank you!