Introduction

This is a report on ongoing joint work with Nuno Romão and Christian Wegner. We are studying problems originating in physics using methods from topology and L^2-cohomology. Roughly speaking, I'm responsible for the topology, Romão for the mathematical physics and Wegener for the L^2 cohomology. But this is only a very rough approximation, since there is massive interaction between the pieces.
For the topology I have had very helpful discussions with Jørgen Tornehave.

What is the motivation?

My work is essentially about algebraic topology of a certain class of configuration spaces. Considered as part of topology, these results my have some mild interest, but the real motivation comes from outside of algebraic topology.. The hope is that these spaces will be closely related to certain moduli spaces of vortices. In the simplest cases, this is more than a hope.

The vortices are solutions to equations similar to but simpler than the Seiberg-Witten equations.

A simple case of a vortex.

In the most basic form, consider a surface with a metric, a compatible complex structure and a complex line bundle.
A vortex is given by a pair, consisting of a section ϕ in the complex line bundle and a connection A on the line bundle. The section is assumed to be flat with respect to the connection. We also assume the following equation

$$*F_A + \mu \circ \phi = 0$$

Here the map μ is given by a choice of moment map $\mathbf{C} \to i\mathbf{R}$. Different choices of moment map gives essentially different sets of solutions.

The case $\Sigma = \mathbf{C}$

Without discussing this side of the story in details, we mention the following theorem by Taubes. Let $\Sigma = \mathbf{R}^2$.
For any choice of n, given any n points (more precisely, an effective divisor of degree n), there is a solution to the vortex equation for which the section ϕ vanishes at exactly the points of the divissor.
This solution is unique up to gauge equivalence.

Vortices on compact manifolds.

We can generalise a little. First, consider the case of the Riemann sphere. The group $\mathbf{C}^*$ acts on S^2 with two fixed points. A section of the bundle over Σ with fiber S^2 determines two divisors on Σ, the inverse images of 0 and ∞. One can show (Baptista) that a pair of divisors determines a unique-up-to-gauge equivalence solution to the generalized vortex equations. (There are some complications here, but let's just ignore those).
The goal of this theory is to say something about moduli spaces of vortices on a surface with values in Kähler manifolds with a $(\mathbf{C}^*)^k$-action - think of toric varieties.
In some cases, this moduli space is homeomorphic to a configuration space, which we will discuss below.

Configuration spaces

For the rest of this talk, we fix an oriented, closed surface Σ. Let $S^k(\Sigma)$ be the symmetric power of Σ. We consider this space as a configuration space of k unordered points on Σ. The unusual thing about this is that we don't demand that the points are distinct. We consider a configuration space consisting of divisors A_i of $r+1$ colors on a surface Σ. We fix the number of points in each set, that is we fix numbers $k_0, \ldots, k_r$ and consider $r+1$ sets $A_i \subset S^{k_i}(\Sigma)$ subject to a further condition.

The sociogram

There is a sociology of colors which we encoded in a graph Γ. The vertices $V(\Gamma)$ of the graph are the colors $0, \ldots, r$. If two colors hate each other, we connect them by an edge. The set of edges of the graph are unsurprisingly called $E(\Gamma)$. We obtain an unoriented graph Γ with no loops and no multiple edges.

Actually there are good reasons to follow the sociologists and instead consider the complementary graph, where two colors are joined by an edge if they do not hate each other. But it seems that for our present purposes, the opposite convention is more convenient.

Space of many colors

We define the configuration space of Σ belonging to the color scheme $(\Gamma, \mathbf{k})$ as the subspace

$$S^{(\Gamma,\mathbf{k})}(\Sigma) \subset \prod_{0 \leq i \leq r} S^{k_i}(\Sigma)$$

determined by the condition that a point

$$(A_0, \ldots A_r).$$

is in $S^{(\Gamma,\mathbf{k})}(\Sigma)$ if and only if

$$X_i \cap X_j = \emptyset \text{ when } (i,j) \text{ is an edge of } \Gamma.$$

Definition of the space

If there is one single color, we obtain the symmetric powers of the surface. If there are $r+1$ points of distinct colors which all hate each other, we get the configuration space of $r+1$ distinct, ordered points. In general we get some kind of interpolation between these two cases.

From now on, suppose that there are $k_i \geq 2$ points of the color indexed by i (The cases where some $k_i = 1$ are somewhat special). We don't like to carry all decorating indexes along, so for now we fix $(\Gamma, \mathbf{k})$ and denote the corresponding space by $X = S^{(\Gamma,\mathbf{k})}(\Sigma)$. There are obvious maps $P_i : X \to S^{k_i}(\Sigma)$, forgetting the points not of color i.

How does the space of rational functions fit in?

The space of rational functions, considered by Segal and others, can be described as the space of pairs of effective divisors (A, B) on S^2 such that $\infty \in A$ and such that A and B are disjoint.
In our setup, we can consider triples of disjoint divisors (X, A', B) such that the pairs (X, B) and (A', B) are disjoint, and such that $X = \{x\}$ has exactly one element. The map $(\{x\}, B, C) \mapsto x \in S^2$ is a fibration, and the fibre over ∞ can be identified with the space of rational functions by the map

$$(X, A', B) \mapsto (X \cup A', B)$$

Braids

We can ask various questions here. The first one would be about the fundamental group. There is a braid group type description of this group: Consider braids of $r+1$ colors on Σ. The braids are allowed to pass through each other, unless they belong to colors that hate each other. This is what we call the divisor braid groups, since each color defines a divisor on Σ.

There is a canonical homomorphism

$$P_* = \prod_i P_{i*} : \pi_1(X) \to \prod_i \pi_1(S^{k_i}(\Sigma)) \cong (\mathbf{Z})^{2g(r+1)}$$

In case $k_i \geq 2$, the kernel will be generated by commutators, so we can identify the map P with the Hurewicz map of the space X.

The fundamental group is metabelian!

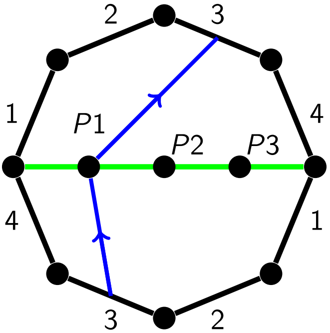
In general, it turns out that the map P_* will have a kernel. But this kernel will be independent of the suface. Still assuming that each $k_i \geq 2$, there is a group $A = A(\Sigma, \mathbf{k})$ only depending on Σ and $\mathbf{k}$ and a central extension

$$0 \to A \to \pi_1(X) \xrightarrow{h} (\mathbf{Z})^{2g(r+1)} \to 0$$

The fundamental braids

We obtain fundamental generators by moving a single point along a path, and keeping the others fixed. If α is the path in Σ which does not touch one any points it hates, we call the correspoding braid $\Phi(\alpha)$. Now, suppose you have two such closed paths, α and β, corresponding to paths of two distinct points of the same color. You can obtain a the two new paths $\Phi(\alpha\beta) = \Phi(\alpha)\Phi(\beta)$ and $\Phi(\beta\alpha) = \Phi(\beta)\Phi(\alpha)$. The corresponding elements of $\pi_1(X)$ are equal.
This means that $\Phi(\alpha)$ only depends on the homology class $[\alpha] \in H_1(\Sigma)$.

This path picks up some homology

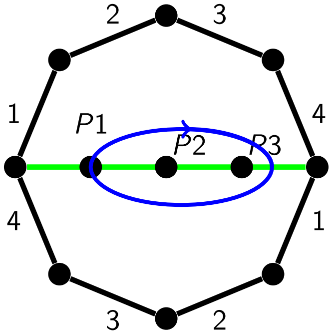


The basic relation

Two generators corresponding to two different colors connected by an edge do not necessarily commute. The commutators of such elements will generate $A(\Sigma; \mathbf{k})$. Lets say that $\alpha_{i,j}$ is the braid obtained by letting one point of color i move once in positive direction around a point of color j.
Here is the basic relation between these classes:

$$\sum_i k_i \alpha_{i,j} = 0$$

One trip around the edge of the world



Generators and relations

Define the following abelian group S. The generators are symbols $\sigma_{i,j}$. The relations are

$$\sigma_{i,j} - \sigma_{j,i} = 0$$
$$\sigma_{i,i} = 0$$
$$\sum k_i \sigma_{i,j} = 0$$

There is a map $\phi : S \to A$ given by $\phi(\sigma_{i,j}) = \alpha_{i,j}$. Using known results about braid groups, it's not so hard to show that this map is surjective. We claim that this map is actually an isomorphism.

The braid considered as a chain

To prove injectivity, we construct a map in the opposite direction $A \to M$. That is, we construct an invariant on A with values in the Abelian group M. To do this, we use a generalization of linking numbers to the colored situation. Let A be a divisor braid in the kernel of h. Each color divisor defines a cycle a_i in $S^1 \times \Sigma$. Since the class a_i is in the kernel of h, it projects to a trivial homology class in Σ. It follows that a_i is homologous to $k_i[S^1]$. Chose a chain A_i such that $\partial A_i = a_i - k_i[S^1]$.

The bounding chain

Given classes $m_{i,j} \in M$ such that

$$\sum_i k_i \otimes m_{i.j} = 0$$

and

$$m_{i,j} = 0 \text{ if } (i,j) \text{ is an edge in } \Gamma.$$

we can define a chain $\theta(A, B)$ by the formula

$$\theta(A, B) = \sum_{i.j} A_i \otimes b_j \otimes m_{i,j}$$

Boundary of θ

We consider this as a 3-chain

$$\theta(A, B) \in C_*(S^1 \times \Sigma) \otimes C_*(S^1 \times \Sigma) \otimes M$$

and note that its boundary is

$$\begin{aligned}
\sum_{i.j} \partial A_i \otimes b_j \otimes m_{i,j} &= \sum_{i.j} (a_i - k_i[S^1]) \otimes b_j \otimes m_{i,j} \\
&= \sum_{i.j} a_i \otimes b_j \otimes m_{i,j} \\
&- \sum_{i.j} [S^1] \otimes b_j \otimes k_i m_{i,j} \\
&= \sum_{i.j} a_i \otimes b_j \otimes m_{i,j}
\end{aligned}$$

Eilenberg and Zilber

So,

$$\partial\theta(A, B) = \sum_{i.j} a_i \otimes b_j \otimes m_{i,j}$$

The sum is actually taken over (i, j) where $e_{i,j}$ is an edge in Γ. Now recall the Eilenberg-Zilber chain homotopy equivalence

$$EZ : C_*(X) \otimes C_*(Y) \to C_*(X \times Y)$$

Applying this, we get that the boundary of $EZ(\theta(A, B))$ is contained in

$$C_*((S^1 \times \Sigma)^2 \setminus \Delta); M)$$

The link invariant

That is, we have defined a cycle

$$\theta(A,B) = [\sum_{i.j} A_i \otimes b_j \otimes m_{i,j}] \in H_3((S^1 \times \Sigma)^2, (S^1 \times \Sigma)^2 \setminus \Delta); M)$$

Here Δ is the diagonal. By Thom isomorphism, this group is isomorphic to M generated by a Thom class of the oriented normal bundle of Δ.
This gives an invariant of divisor links which is invariant under homotopy of the links.

The linking invariant of a braid

The quadratic form $\theta(A, A)$ of this gives a homomorphism $A \to S$, and computation shows that this a homomorphism and an inverse of the surjective map $S \to A$.
This computes the fundamental groups of X in the sense that we have given a presentation in generators and relations. Now, we could ask the following follow up question: What happens if we fix the graph Γ, and vary the numbers k_i? For each graph this gives a function from vectors of natural numbers to abelian groups.

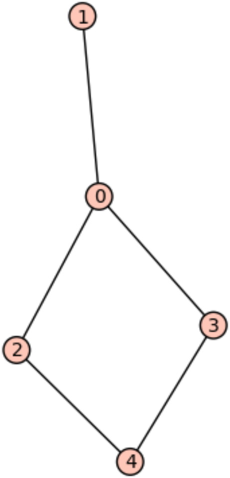
The free part

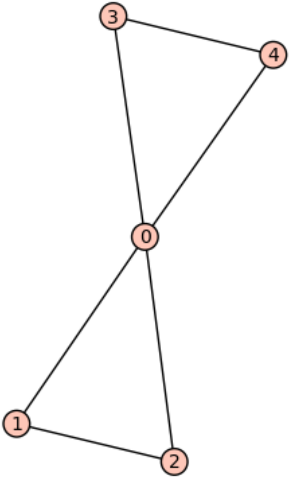
Here is the easy part.

Theorem

If Γ is not a bipartite graph, the rank of the torsion free part of $A(\Gamma, \mathbf{k})$ is # Edges of Γ - # Vertices of Γ.
If Γ is a bipartite graph, the rank of the torsion free part of $A(\Gamma, \mathbf{k})$ is # Edges of Γ - # Vertices of Γ+1.

Bipartite and not





Proof of theorem, part 1

Here is a proof. The group A is generated by the symbols $\sigma_{i,j}$ for $i < j$, with the relations $P_j = \sum k_i \sigma_{i,j}$ (where we interprete $\sigma_{i,j} = \sigma_{j.i}$ for $i > j$). This gives a map $P : \mathbf{Z}^{|V|} \to \mathbf{Z}^{|E|}$, where V, E are the vertices and edges of Γ. The group A is the cokernel of P, so that

$$\mathrm{Rank}(A) = (\#\text{ Edges}) - (\#\text{ Vertices}) + \mathrm{Rank}\,\ker(P)$$

So we have to compute the rank of P.

Proof of theorem, the end

The kernel of the map P consists of formal sums

$$u = \sum x_v v$$

satisfying that is $e \in E$ is an edge between v and w, then $k_v x_w + k_w x_v = 0$. That is, the coefficients x_v are equal up to sign. The sign of a vertex determines a bi-partitioning of Γ. If Γ is bipartite, the kernel of P has rank 1, and if Γ is not bi-partite, it has rank 0.

Tricky torsion

It's more difficult to determine the torsion subgroup of A.
We construct torsion elements in A. Assume that $\Gamma' \subset \Gamma$ is a bipartite full subgraph (full means that it's determined by it's vertices). Let $c = c(\Gamma')$ be the the greatest common divisor is over $v \in V(\Gamma')$. Consider the formal sum of vertices

$$F(\Gamma') = \frac{1}{c} \sum_{v \in V(\Gamma')} \pm k_v v$$

where the sign is given by the bipartitioning. $F(\Gamma')$ is not divisible, and $P(F(\Gamma')) \in \mathbf{Z}[E(\Gamma)]$ determines the trivial element in A.

The case of two colors

To see what the torsion might look like, consider the case $r = 1$. There are two vertices v, w and one edge e. The group A is generated by the class e, and there are two relations:

$$P_v = k_w e = 0$$
$$P_w = k_v e = 0$$

This immediately gives that $A \cong \mathbf{Z}/GCD(k_v, k_w)$
In general, a formula for A typically involves products and quotients of the greatest common divisors of various subsets of $\{k_i\}$

The min-plus ring

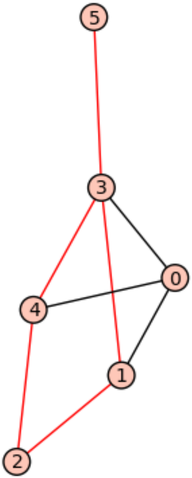
If we are interested in the p-torsion of A, it is sufficient to consider the case when $k_i = p^{e_i}$. The operations of taking products and GCD of such numbers is related to the max-plus structure of the integers:

$$GCD(p^e, p^f) = p^{\min(e,f)}$$
$$p^e p^f = p^{e+f}$$

It seems reasonable to expect that one can give the structure of the p-torsion of A in terms of this semi-ring structure, which brings us close to tropical rational functions. (On some days I think that I have a proof of this).

The fundamental class of Γ'

If $P(F(\Gamma'))$ is divisible by an integer n, so that $P([\Gamma']) = n[\Gamma'] \in \mathbf{Z}[E]$, then the image of $[\Gamma']$ in A is an n-torsion element.



Here we have the inclusion of a bipartite graph $\Gamma' \subset \Gamma$. The class $P([\Gamma'])$ is a formal sum of the three edges not belonging to Γ'. Precisely, it will be $k_0 k_3 e_{0,3} + k_0 k_4 e_{0,4} + k_0 k_1 e_{0,1}$. If k_0 is big enough, we obtain a non-trivial torsion class.

2-torsion is slightly special

There are some more 2-torsion classes. Take any subgraph $\Gamma' \subset \Gamma$. Even if Γ is not bi-partite, we can form

$$[\Gamma'] = \sum_{v \in V(\Gamma')} v$$

If $P([\Gamma'])$ is divisible by 2, we get a 2-torsion class in A. This makes the 2-torsion slightly different from the odd torsion. The exact pertubation can be figured out, but we won't go into details. For ease of exposition, we will concentrate exclusively on odd torsion in the sequel. (Don't worry, there are only a few slides left now).

An abstract view

Now we want to determine the odd torsion of $A(\Sigma, \mathbf{k})$. By general theory of finite Abelian groups, it suffices if we for each odd prime p and every n can find the subgroup of elements in A annihilated by p^n. We fix p, and for any integer n, we let $v_p(n)$ be the p-valuation of n, so that $n = p^{v_p(n)} n'$ where n is an integer, relatively prime to p.

Sudden appearance of a herd of fundamental classes

Let Δ be a connected, bipartite subgraph of Γ. We consider the following number.

$$M(\Delta) = \min_{e} v_p(k_v k_w)$$

where the minimum is computed over all e connecting two vertices $v \in V(\Delta)$ and $w \notin \Delta$. We also consider

$$m(\Delta) = \min_{v \in V(\Delta)} v_p(k_v)$$

In general, $m(\Delta) \leq M(\Delta)$. The fundamental class $[\Delta]$ is a torsion class of order p^n where $d = d(\Delta) = M(\Delta) - m(\Delta)$.

Chosing suitable subgraphs

We can describe a set of bi-partite graphs Δ such that the classes $[\Delta]$ with $d(\Delta) = d$ generate the p^n torsion of $A(\Sigma, \mathbf{k})$, and further that they are independent over $\mathbf{Z}/p$ modulo the p^{n-1} torsion of A. For every pair d, m, consider the set $\mathcal{B}_{d,m}$ of bipartite graphs Δ such that $m(\Delta) = m$ and $d(\Delta) = d$. We say that such a graph is primitive if it doesn't contain any proper subgraph in $\mathcal{B}_{d,m}$. The set of all primitive graphs will do for the purpose above.