$\ell^{q,1}$ cohomology in Heisenberg groups

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Sobolev Inequality

Sobolev Inequality in $\mathbb{R}^n$

Given $1 \leq p < n$, and $q \in \mathbb{N}$ as $\frac{1}{p} - \frac{1}{q} = \frac{1}{n}$, then if $u \in C_0^\infty(\mathbb{R}^n)$, we have

$$\|u\|_q \leq C(n,p)\,\|du\|_p \tag{1}$$

The most important of the Sobolev inequalities in (1), is the one where $p = 1$:

- it implies the others;
- it is equivalent to the isoperimetric inequality.

We will focus on a generalisations of such inequalities.

$L^{q,p}$ cohomology

Definition

Given X a manifold, define

$$L^{q,p}H^k(X) = \{L^p \text{ closed } k\text{-forms } \omega\}/d\{L^q(k-1)\text{-forms } \phi \text{ s.t. } d\phi \in L^p\}.$$

Question:

Why are we considering $L^{q,p}$ forms?

$\ell^{q,p}$ cohomology

Theorem by Pansu and Rumin (2018)

Given X a contractible Lie group, then

$$\ell^{q,p}H^k(X) = L^{q,p}H^k(X)$$

where $\ell^{q,p}$ is cohomology of simplicial complexes.

Definition

Given X a symplicial complex. Cochains are functions on symplices, then

$$\ell^{q,p}H^k(X) = \{k\text{-cocycles in } \ell^p\}/d\{(k-1)\text{-cochains in } \ell^q\}\,.$$

- by definition, the $\ell^{q,p}$ cohomology is a quasi-isometry invariant

$L^{q,p}$ cohomology in $\mathbb{R}^n$

Proposition

$X = \mathbb{R}^n$, then $L^{q,p}H^k(X) = 0$ if $1 < p \leq q < \infty$ and $\frac{1}{p} - \frac{1}{q} = \frac{1}{n}$

Proof

- Laplace-Beltrami $\Delta = d^*d + dd^*$ has a pseudo-differential inverse that commutes with d;
- $T := d^*\Delta^{-1}$ has a homogeneous kernel of degree $1 - n$
- $T : L^p \to W^{1,q}$ is bounded provided $\frac{1}{p} - \frac{1}{q} = \frac{1}{n}$ (Calderon-Zygmund 1952)
- Finally $Id = dT + Td$

$L^{q,p}$ cohomology in Carnot groups

We need a homogeneous Laplacian

Use homogeneous groups, and in particular Carnot groups, where

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_s \, , \delta_t = t^i \text{ on } \mathfrak{g}_i$$

Kohn's Laplacian on functions

For a function $u \in \Omega^0$, we define

$$d_c u := du\big|_{\mathfrak{g}_1} \text{ and } \Delta := d_c^* d_c \, .$$

This is homogeneous of degree 2 under Carnot dilations δ_t.

In general on Ω^k

In general, given $\alpha \in \Omega^k$, there is no differential homogeneous Laplacian on α, since left-invariant forms split under δ_t into several weight spaces.

Easiest example: Heisenberg groups $\mathbb{H}^{2n+1}$

On the Heisenberg groups, any k-form with $0 < k < 2n+1$ has two different weights, k and $k+1$:

$$\Lambda^k \mathfrak{g} = \Lambda^k \mathfrak{g}_1 \oplus \Lambda^{k-1} \mathfrak{g}_1 \otimes \Lambda^1 \mathfrak{g}_2$$

Possible homogeneous Laplacian

A pseudodifferential homogeneous Laplacian

- let $|\nabla| = \Delta^{1/2}$;
- let $|\nabla|^N$ be the operator acting componentwise, which is $|\nabla|^w$ on forms of weight w
- $d^\nabla := |\nabla|^{-N} d |\nabla|^N$ pseudodifferential operator of order 0, hence
- $\Delta^\nabla := (d^\nabla)^* d^\nabla + d^\nabla (d^\nabla)^*$ is a homogeneous Laplacian of order 0

- Δ^∇ admits a pseudodifferential inverse (Helffer-Nourrigat 1979, Christ-Geller-Glowacki-Polin 1992), hence d^∇ admits a homotopy
- let $K^\nabla := (d^\nabla)^*(\Delta^\nabla)^{-1}$, we have $1 = d^\nabla K^\nabla + K^\nabla d^\nabla$
- hence $K := |\nabla|^N K^\nabla |\nabla|^{-N}$ is a homogeneous homotopy for d,
- K^∇ is bounded on L^p (Folland 1975), hence K is bounded on

$$L^{N,p} := \{\alpha\,;\, |\nabla|^{-N}\alpha \in L^p\},$$

and on $L^{N-m,p}$ for every constant m.

Proposition

$$\ell^{q,p}H^k(G) = 0 \text{ if } 1 < p, q < \infty,\ \tfrac{1}{p} - \tfrac{1}{q} \geq \tfrac{b-a}{Q},$$

- b is the maximal weight in degree k,
- a is the minimal weight in degree $k-1$,
- $Q = \sum_{i=1}^{s} i \cdot dim(V_i)$.

In the Heisenberg groups $\mathbb{H}^{2n+1}$

For any $0 < k < 2n+1$

$$b = max\{k, k+1\} = k+1 \text{ and } a = min\{k-1, k\} = k-1$$

which means

$$\frac{b-a}{Q} = \frac{2}{2n+2} = \frac{1}{n+1} \qquad \text{NOT SHARP!}$$

Rumin complex

Strategy

Replace (Ω^*, d) with a subcomplex $(\mathcal{E}_0^*, d_c)$ that

- has forms of fewer weights;
- is homotopic to the de Rham complex

Weight of a 1-form $\alpha \in \Lambda^1 \mathfrak{g}$

$$\alpha \text{ has (pure) weight } p \text{, i.e. } w(\alpha) = p \iff \alpha^\natural \in V_p$$

Weight of an h-form $\beta \in \Lambda^h \mathfrak{g}$

$$\beta \text{ has (pure) weight } p \text{, i.e. } w(\beta) = p \iff \beta = \text{lin. comb. of } h\text{-form}$$
$$\theta_{i_1} \wedge \cdots \wedge \theta_{i_h} \text{ s.t. } w(\theta_{i_1}) + \cdots + w(\theta_{i_h}) = p.$$

Rumin Complex

$d = d_0 + d_1 + \cdots + d_k$

Let $\alpha = \sum_i f_i \, \theta_i^h$ be an arbitrary h-form of pure weight p as before, then we can write:

$$d\alpha = d_0\alpha + d_1\alpha + \cdots + d_k\alpha$$

$\mathbf{d_i}$ is the part of d which **increases the weight** of the form α by **i**:

- $d_0\alpha = \sum_i f_i \, d\theta_i^h \ \in \Omega^{h+1,p}$;
- $d_l\alpha = \sum_i \sum_{X_j \in V_l} X_j(f_i) \, \theta_j \wedge \theta_i^h \ \in \Omega^{h+1,p+l}$ for $l \in \{1, \ldots, k\}$.

The operator d_0 is an algebraic operator: it involves no differentiation and does not increase the weight.

Rumin Complex

Proposition

Two h-covectors α, $\beta \in \Lambda^h \mathfrak{g}$ with $w(\alpha) \neq w(\beta)$ are orthogonal.

$d_0^2 = 0$

Let us write down d explicitly according to the weight increase and let $\alpha = f\,\theta_j^h \in \Omega^{h,p}$, then

$$
\begin{aligned}
0 = d^2\alpha &= d(d_0\alpha + d_1\alpha + \cdots + d_k\alpha) \\
&= (d_0 + d_1 + \cdots + d_k)(d_0\alpha + d_1\alpha + \cdots + d_k\alpha) \\
&= \underbrace{d_0^2\alpha}_{\text{weight p}} + \underbrace{d_0 d_1\alpha + d_1 d_0\alpha}_{\text{weight p+1}} + \underbrace{(d_0 d_2 + d_1 d_1 + d_2 d_0)\alpha}_{\text{weight p+2}} + \cdots
\end{aligned}
$$

Rumin Complex

Corollary

(Ω^*, d_0) is a complex, so we define

$$\mathcal{E}_0^h := Ker\, d_0 \cap \left(Im\, d_0\right)^{\perp} \subset \Omega^h\,.$$

$$\overbrace{\mathrm{Im}\,\mathrm{d}_0 \;+\; \mathcal{E}_0}^{\mathrm{Ker}\,\mathrm{d}_0} \;+\; \left(\mathrm{Ker}\,\mathrm{d}_0\right)^{\perp}$$

$$\downarrow \qquad \downarrow \qquad\qquad \downarrow \qquad \Leftarrow \text{This isomorphism defines } \mathrm{d}_0^{-1}$$

$$0 \qquad 0 \qquad \mathrm{Im}\,\mathrm{d}_0 \;+\; \mathcal{E}_0 \;+\; \left(\mathrm{Ker}\,\mathrm{d}_0\right)^{\perp}$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$0 \qquad 0 \qquad \mathrm{Im}\,\mathrm{d}_0$$

Constructing the Rumin complex

$$
\begin{array}{ccc}
\Omega^* & \xrightarrow{\quad \mathrm{d} \quad} & \Omega^* \\
\Pi_E \downarrow \uparrow \iota & & \Pi_E \downarrow \uparrow \iota \\
E^* & \xrightarrow{\quad \mathrm{d} \quad} & E^* \\
\Pi_{\mathcal{E}_0} \downarrow \uparrow \Pi_E & & \Pi_{\mathcal{E}_0} \downarrow \uparrow \Pi_E \\
\mathcal{E}_0^* & \xrightarrow{\quad \mathrm{d}_C \quad} & \mathcal{E}_0^*
\end{array}
$$

Rumin complex of Heisenberg groups

3-dimensional Heisenberg group

- $\mathcal{E}_0^1 = \Omega^{1,1}$ horizontal 1-forms;
- $d_c\colon \mathcal{E}_0^0 \to \mathcal{E}_0^1$ is the horizontal gradient;
- $\mathcal{E}_0^2 = \Omega^{2,3}$ vertical 2-forms;
- $d_c\colon \mathcal{E}_0^1 \to \mathcal{E}_0^2$ has order 2

Heisenberg groups $\mathbb{H}^{2n+1}$

- $\mathcal{E}_0^k \subset \Omega^{k,k}$ (subspace of horizontal forms) for $k \leq n$;
- $\mathcal{E}_0^k \subset \Omega^{k,k+1}$ (subspace of vertical forms) for $k \geq n+1$;
- $d_c\colon \mathcal{E}_0^n \to \mathcal{E}_0^{n+1}$ has order 2, and
- $d_c\colon \mathcal{E}_0^k \to \mathcal{E}_0^{k+1}$ has order 1 otherwise ($k \neq n$).

$\ell^{q,p}$ of Carnot groups

Pansu-Rumin

Let G be a Carnot group of homogeneous dimension Q. Let $[a, b]$ be the scope of weights in Rumin k-forms, let $[a', b']$ be the scope of weights in Rumin $(k-1)$-forms, then

- $\ell^{q,p}H^k(G) = 0$ provided $1 < p, q < \infty$ and
$$\frac{1}{p} - \frac{1}{q} \geq \frac{b - a'}{Q}.$$
- $\ell^{q,p}H^k(G) \neq 0$ if $1 \leq p, q \leq \infty$,
$$\frac{1}{p} - \frac{1}{q} < \frac{\max\{1, b - a'\}}{Q}.$$

$\ell^{q,p}$ cohomology in the Heisenberg group, $p > 1$

In Heisenberg groups $\mathbb{H}^{2n+1}$

- if $k \neq n+1$, $b - a' = 1$, so $\ell^{q,p} H^k(\mathbb{H}^{2n+1}) = 0$ for

$$\frac{1}{p} - \frac{1}{q} \geq \frac{1}{2n+2} \quad \text{with } p > 1$$

- if $k = n+1$, $b - a' = 2$, so $\ell^{q,p} H^{n+1}(\mathbb{H}^{2n+1}) = 0$ for

$$\frac{1}{p} - \frac{1}{q} \geq \frac{2}{2n+2} = \frac{1}{n+1} \quad \text{with } p > 1$$

How to tackle $p = 1$ in $\mathbb{R}^n$?

Theorem (Bourgain-Brezis-Mironescu 2004)

Given ω a closed $(n-1)$-form with compact support in $\mathbb{R}^n$. Then for any 1-form α such that $\nabla\alpha \in L^n$,

$$\left| \int_{\mathbb{R}^n} \alpha \wedge \omega \right| \leq C \|\omega\|_1 \, \|\nabla \alpha\|_n \, .$$

Definition

The averaging map of a k-form ω in L^1 is given by

$$\int_{\mathbb{R}^n} \beta \wedge \omega \text{ where } \beta \text{ has constant coefficient.}$$

How to tackle $p = 1$ in $\mathbb{R}^n$?

Proposition 1 (Corollary of BBM)

$d^*\Delta^{-1}\colon L^1 \to L^{n/n-1}$ is bounded on closed $(n-1)$-forms with vanishing averaging map.

Proposition 2

Given ω a closed $(n-1)$-form in L^1, then ω has vanishing averaging map.

Corollary (Bourgain-Brezis 2007)

$L^{n/n-1,1}H^{n-1}(\mathbb{R}^n) = 0.$

How to tackle $p = 1$ in $\mathbb{R}^n$?

Proposition 3 (Lanzani-Stein 2005)

The BBM inequality for closed k-forms ω with compact support

$$\left| \int_{\mathbb{R}^n} \alpha \wedge \omega \right| \leq C \|\omega_1\| \, \|\nabla \alpha\|_n$$

holds for all $k \leq n - 1$ (but not for $k = n$).

Corollary

$L^{n/n-1,1} H^k(\mathbb{R}^n) = 0$ for $k \leq n - 1$.

$p = 1$ in the Heisenberg groups $\mathbb{H}^{2n+1}$

Homogeneous Laplacian in $\mathbb{H}^{2n+1}$ (Baldi, Franchi, Pansu, 2019)

- $k \neq n, n+1$: $d_c^* d_c + d_c d_c^*$ (order 2);
- $k = n$: $d_c^* d_c + (d_c d_c^*)^2$ (order 4);
- $k = n+1$: $(d_c^* d_c)^2 + d_C d_C^*$ (order 4).

Generalisation of BBM inequality

Proved for Carnot groups by Chanillo-Van Schaftingen in 2008

$p = 1$ in the Heisenberg groups $\mathbb{H}^{2n+1}$

Averaging map vanishes (Pansu, T, 2019)

- The averaging map of a d_c-closed L^1 form ω is given by $\int_{\mathbb{H}^{2n+1}} \omega \wedge \beta$, β left-invariant Rumin form;
- The averaging map is well defined as a map $\ell^{q,1} H^k(G) \otimes H^{n-k}(\mathfrak{g}) \to \mathbb{R}$, with $q = Q/Q-2$ if $k = n+1$ and $q = Q/Q-1$ otherwise;
- The averaging map of d_c-closed L^1 forms of degree less than $2n+1$ vanish.

$p = 1$ in the Heisenberg groups $\mathbb{H}^{2n+1}$

Final result (Baldi, Franchi, Pansu + Pansu, T.)

$\ell^{q,1}H^k(\mathbb{H}^{2n+1}) = 0$ in the following cases:

- if $k \neq n+1$, $1 - \frac{1}{q} \geq \frac{1}{Q}$;
- if $k = n+1$, $1 - \frac{1}{q} \geq \frac{2}{Q}$.

Remark

This is sharp by Pansu-Rumin (2018)