

# Stability of the solitary waves to the KdV equation.

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Mini-course

June 2018.

①

## I. The KdV equation.

→ In 1834, S. Russell engineer for the committee of works of the "British Association for the Advancement of Science" observed a long crested propagating on the surface of the Glasgow-Edinburgh canal along several miles.

→ Together with Robinson, he was able to reproduce this kind of waves in a lab

Their experiments were published in 1844. He called these waves:

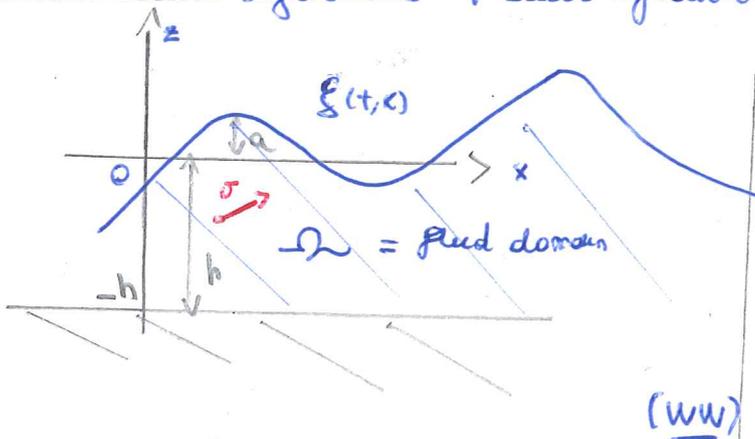
great waves of translation or solitary waves

empirical relation:

$$c^2 = \frac{2}{3} (h+a)$$

$c$  = speed of the wave,  $h$  = depth,  $a$  = height

→ Several mathematicians tried to explain these waves from the water waves equations (Euler equations + free surface)



### water waves equations (Lagrange 1781)

(1) Mass conservation

$$\rho_t + \text{div}(\rho \vec{v}) = 0 \Rightarrow \text{div}(\vec{v}) = 0$$

incompressible fluid

(2) Newton Law

$$\rho_t + v \cdot \nabla v = -\frac{1}{\rho} \nabla P + g$$

(3) Boundary conditions at the free surface

$$\nabla \xi = \left( \frac{\partial \xi}{\partial x} \right) \cdot \vec{v}, P = 0 \text{ at } z = \xi(t, x)$$

(4) Boundary conditions at the bottom

$$\partial_z \vec{v} = 0 \text{ at } z = -h$$

Rg

① (1)+(2) = Euler equations

one of the oldest PDE to be written down (1755) and one of the most difficult to solve

→ connection with the Navier-Stokes equations and the 1 million dollar problem!

② Since it is (still) but even more at the time) possible to solve this equation, the idea is to "zoom on a special region" and derive simpler asymptotic models (see Lagrange in "mémoire sur la théorie du mouvement des fluides" in 1781). It corresponds to derive an asymptotic model from (WW) in a special regime



## II - Solitary waves of KdV

From now on, we work on the renormalized equation:  $\begin{cases} X \rightarrow X-t \\ \varepsilon = 1 \\ \mu = \lambda \varepsilon(\beta) \end{cases}$

$$\text{(KdV)} \quad \partial_t u + \partial_x^3 u + \partial_x(u^2) = 0, \quad x \in \mathbb{R}, t \in \mathbb{R}, u = u(t, x) \in \mathbb{R}$$

We look for special solutions of KdV on the form  $u(t, x) = \phi(x-ct)$   
this is the form of a solitary wave.  
 $\begin{cases} \phi(x) \rightarrow 0 \\ |x| \rightarrow +\infty \end{cases}$   
 $\phi \in C^2(\mathbb{R})$

For  $u$  to be a solution of KdV,  $\phi$  must satisfy

$$-c\phi' + \phi''' + (\phi^2)' = 0.$$

By integrating one time, we get (and using the decay condition  $\phi(x) \rightarrow 0$  as  $|x| \rightarrow +\infty$ )

$$\text{(E)} \quad -\phi'' + c\phi - \phi^2 = 0, \quad x \in \mathbb{R}$$

elliptic  
~~nonlinear~~ **ODE**

Prop 1. Let  $\phi \in C^2(\mathbb{R})$  such that  $\phi(x), \phi'(x) \rightarrow 0$  as  $|x| \rightarrow +\infty$  be a solution of (E).

$$\lfloor \text{Then } c > 0 \text{ and } \int_{\mathbb{R}} \phi^3 dx > 0$$

In particular the solitary waves are evolving to the right (while the dispersion is evolving to the left)!

Proof The proof is based on Pohozaev identities.

① Multiply (E) by  $\phi$  and integrate:

$$\Rightarrow - \int \underbrace{\phi''\phi}_{(\phi')^2} + c \int \phi^2 - \int \phi^3 = 0 \Rightarrow \int (\phi')^2 + c \int \phi^2 = \int \phi^3 \quad (\perp)$$

② Multiply (E) by  $x\phi'$  and integrate:

$$\Rightarrow - \int \phi'' x \phi' + c \int \phi x \phi' = \int \phi^2 x \phi'$$

$$\therefore - \frac{1}{2} \int [(\phi')^2]' x + \frac{c}{2} \int (\phi^2)' x = \frac{1}{3} \int (\phi^3)' x$$

$$\Rightarrow \boxed{\int (\phi')^2 - c \int \phi^2 = -\frac{2}{3} \int \phi^3} \quad (2) \quad \text{Pohozeev identity}$$

③ Conclusion

$$(1) + \frac{3}{2}(2) \Rightarrow \frac{5}{2} \int (\phi')^2 = \frac{c}{2} \int \phi^2 \Rightarrow \boxed{c > 0}$$

$$(1) - (2) \Rightarrow 2c \int \phi^2 = \underbrace{\left(1 + \frac{2}{3}\right)}_{\frac{5}{3}} \int \phi^3 \Rightarrow \boxed{\int \phi^3 > 0}$$

Integrating (E) (left as an exercise)

$$\text{Multiply (E) by } \phi' \Rightarrow -\frac{1}{2} [(\phi')^2]' + \frac{c}{2} [\phi^2]' - \frac{1}{3} [\phi^3]' = 0.$$

$$\text{integrating (and using } \int \phi' \phi \rightarrow c \text{ as } |x| \rightarrow \infty) \Rightarrow -\frac{1}{2} (\phi')^2 + \frac{c}{2} \phi^2 - \frac{1}{3} \phi^3 = 0.$$

$$\text{Thus } \boxed{(\phi')^2 = c \phi^2 - \frac{2}{3} \phi^3 > 0}$$

$$\Rightarrow \phi' = \frac{d\phi}{dx} = \pm \sqrt{c \phi^2 - \frac{2}{3} \phi^3}$$

$$\Rightarrow \int_{\phi_0}^{\phi} \frac{d\phi}{\sqrt{c \phi^2 - \frac{2}{3} \phi^3}} = \pm \int_{x_0}^x dx = \pm (x - x_0)$$

$$\Rightarrow \int_{\phi_0}^{\phi} \frac{d\phi}{\sqrt{c \phi^2 - \frac{2}{3} \phi^3}} = \pm \sqrt{c} \int dx$$

④ the change of variable  $\int \phi = \frac{3}{2} c \operatorname{sech}^2(u) = \frac{3}{2} c \cdot \frac{1}{\operatorname{ch}^2(u)}$

$$\int d\phi = -2 \frac{3}{2} c \frac{\operatorname{sh}(u)}{\operatorname{ch}^3(u)} du$$

Thus

$$\int \frac{d\phi}{\sqrt{\phi^2 - \frac{3}{2}\phi^3}} = -\lambda \int \frac{\frac{3}{\lambda}c \operatorname{th}(u) \operatorname{sech}^2(u) du}{\frac{3}{\lambda}c \operatorname{sech}^2(u) \sqrt{1 - \operatorname{sech}^2(u)}}$$

$$= -\lambda \int \frac{\operatorname{th}(u)}{\sqrt{\frac{c^2 \operatorname{th}^2(u) - 1}{c^2 \operatorname{th}^2(u)}}} du = -x \int \frac{\operatorname{th}(u)}{\sqrt{\operatorname{th}^2(u)}} du = \mp x \int du = \pm c \int dx.$$

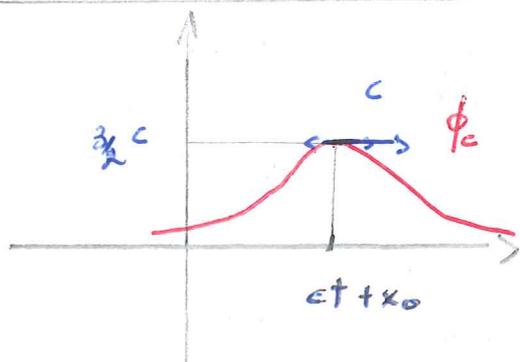
$$u \pm \frac{\sqrt{c}}{\lambda} (x - x_0).$$

Finally we deduce that  $\phi_c(x) := \frac{3}{\lambda}c \operatorname{sech}\left(\frac{\sqrt{c}}{\lambda}(x - x_0)\right)$  is a solution to (E).  $\square$

it is the unique (up to translation) solution to (E).

Conclusion: We found a family of solutions of (KdV)

$$\left[ \phi_{c, x_0}(t, x) := \frac{3}{\lambda}c \operatorname{sech}\left(\frac{\sqrt{c}}{\lambda}(x - ct - x_0)\right) \right]; \begin{cases} c > 0 \\ x_0 \in \mathbb{R} \end{cases}$$



Obs: ①  $\phi_c$  has exponential decay at infinity

② the height of  $\phi_c$  is proportional to its speed (as in tunnel experiments)!

Computation.

$$\|\phi_{c, x_0}\|_{L^2} = \frac{3}{\lambda}c \left\{ \int \operatorname{sech}^4\left(\frac{\sqrt{c}}{\lambda}(x - ct - x_0)\right) dx \right\}^{1/2} =$$

$$= c^{1-1/4} \left\{ \frac{3}{2} \int \operatorname{sech}^2(y) dy \right\}^{1/2} = c^{3/4} \underbrace{\|\phi_{1, 0}\|_{L^2}}_{\text{cte.}}$$

$$\begin{cases} y = \frac{\sqrt{c}}{\lambda}(x - ct - x_0) \\ dy = \frac{\sqrt{c}}{\lambda} dx \end{cases}$$

In other words these waves can have any arbitrary size in  $L^2$  (by changing  $c$ ).

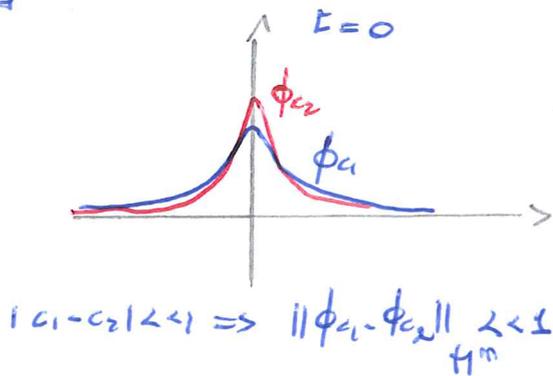
### III - Stability of solitary waves.

Question: are these waves observable in nature? we expect that they are.

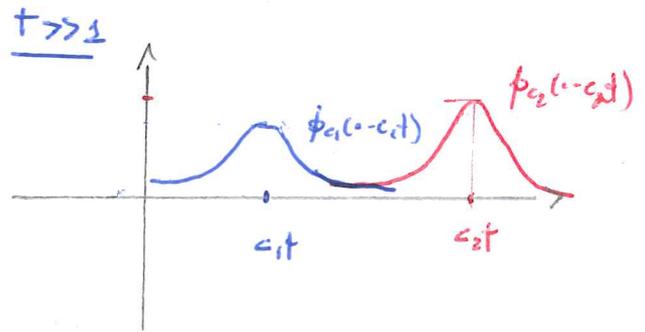
↳ thus raise the question of stability.

↳ Intuitively, if  $u_0$  is close to  $\phi_c$ , then the solution  $u(x,t)$  emanating from  $u_0$  remain close to  $\phi_c(x-ct)$  for all time  $t \geq 0$ .

Rq: this cannot be true: take  $\phi_{c_1}$  and  $\phi_{c_2}$  with  $|c_1 - c_2|$  small



~>



if  $t \gg 1$  large enough, then

$$\|\phi_{c_1}(\cdot - c_1t) - \phi_{c_2}(\cdot - c_2t)\|_{H^0}^2 \sim \|\phi_{c_1}(\cdot - c_1t)\|_{H^0}^2 + \|\phi_{c_2}(\cdot - c_2t)\|_{H^0}^2 \geq 1.$$

For this reason, we need to refine the definition

Def [Orbital stability or stability in shape]

Let  $\epsilon > 0, \exists \delta > 0$  s.t.  $\|u_0 - \phi_c\|_{H^m} < \delta \Rightarrow \int_{\mathbb{R}} \inf_{\sigma \in \mathbb{R}} \|u(\cdot, t) - \phi_c(x - ct - \sigma)\|_{H^1} < \epsilon$   
 where  $u(t, \cdot)$  is the solution evolving from  $u_0$  at time  $t=0$   $\forall t \in \mathbb{R}$ .

Rq For this definition to make sense, we should be able to talk about the solution of KdV corresponding to an initial data  $u_0 \in H^m(\mathbb{R})$  in the same space  $H^m(\mathbb{R})$  for all time  $t \geq 0$ .

↳ we need a global (in time) well-posedness theory for KdV in a suitable function space

Conserved quantities by the flow of KdV

Mass:  $\frac{1}{2} \int_{\mathbb{R}} u(t,x)^2 dx = \frac{1}{2} \int_{\mathbb{R}} u(0,x)^2 dx, \forall t \in \mathbb{R}$   
 " " " "  $\mathcal{M}[u](t) \quad \mathcal{M}[u](0)$

Energy  $E[u](t) = \frac{1}{2} \int_{\mathbb{R}} (u_x(t,x))^2 dx - \frac{1}{3} \int_{\mathbb{R}} u(t,x)^3 dx = E[u](0), \forall t \in \mathbb{R}$

if  $u$  is a solution of KdV

Exercise: verify these identities for smooth decaying functions.

Remark: these quantities are well defined in the space

$$H^1(\mathbb{R}) = \left\{ f : \int_{\mathbb{R}} f^2 dx + \int_{\mathbb{R}} (f')^2 dx < \infty \right\}$$

$H^1(\mathbb{R})$  is a Hilbert space with the norm  $\|f\|_{H^1} = \left\{ \int_{\mathbb{R}} f^2 dx + \int_{\mathbb{R}} (f')^2 dx \right\}^{\frac{1}{2}}$

Theo [Keng, Ponce, Vega (1983)] unique (in a certain sense)

Let  $u_0 \in H^1(\mathbb{R})$ . Then there exists a  $\checkmark$  solution  $u \in C^0(\mathbb{R}; H^1(\mathbb{R}))$  of KdV which ~~is~~ such that  $u(0,x) = u_0(x)$ .

Moreover, the flow map:  $u_0 \in H^1(\mathbb{R}) \mapsto u \in C^0(\mathbb{R}; H^1(\mathbb{R}))$  is continuous (where  $\mathcal{D} = \mathcal{D}(u_0, \tau) \supset \mathbb{R}$ ).

In other words, the initial value problem associated to KdV is globally well posed in the energy space  $H^1(\mathbb{R})$ .

- Remarks
- ① The proof is involved and uses deep tools from Harmonic Analysis
  - ② It relies on a deep idea of T. Kato (1986), Kuznetsov-Tomunskii (1981)
  - ③ The result was improved to lower regularities by

Bourgain ( $L^2(\mathbb{R})$ , 1984), Keng, Ponce, Vega ( $H^s(\mathbb{R})$ ,  $s > -3/4$ , 1986), Killip-Visen ( $H^1(\mathbb{R})$ , 2018)

Theorem [Benjamin 1972, Bona 1975, Weinstein 1985]

The solitary waves associated to KdV are orbitally stable in the energy space  $H^1(\mathbb{R})$ .

In other words: given  $c > 0$ , ~~error~~,  $\exists \delta > 0, C_0 > 0$  st  $\forall \delta \in (0, \delta_0)$

$$\forall u_0 \in H^1(\mathbb{R}), \text{ if } \|u_0 - \phi_c\|_{H^1} < \delta \Rightarrow \begin{cases} \|u(t, \cdot) - \phi_c(\cdot - \sigma(t))\|_{H^1} < C_0 \delta \\ \forall t \geq 0 \text{ for a } c^\perp \text{ function } \sigma(t) \\ \text{satisfying } |\sigma'(t) - c| < C_0 \delta. \end{cases}$$

Remarks ① The proof of Benjamin was completed by Bona  
The ideas lie in the work of Boussinesq (1872).

② Other proofs have been given by Bona, Sogge, Strauss (1987)  
Grillakis, Shabat, Strauss (1987), Weinstein (1985).

~~How we go~~

#### IV - Proof of the stability theorem.

How we follow the proof of Weinstein which provides the additional information on  $\sigma(t)$ . (see also the nice survey of C. D'Amore (2015)).

First step Fix  $c = 1$ ,  $\phi := \phi_1$  (the other cases can be obtained by scaling)

The idea (which already appears in Boussinesq) is to consider the "action" functional  $F[u] := E[u] + c \Pi[u]$ . *Lyapunov functional*

$F$  is conserved along the flow of KdV (i.e.  $F[u](t) = F[u_0] \forall t \in \mathbb{R}$ )

Second step write  $u(t, x) = \phi(x - \sigma(t)) + \varepsilon(t, x)$

$$\text{Then } \Pi[\phi + \varepsilon] = \frac{1}{2} \int \phi^2 + \int \phi \varepsilon + \frac{1}{2} \int \varepsilon^2$$

$$E[\phi + \varepsilon] = \frac{1}{2} \int [(\phi + \varepsilon)_x]^2 - \frac{1}{3} \int (\phi + \varepsilon)^3$$

$$= \frac{1}{2} \int (\phi')^2 + \int \phi' \varepsilon_x + \frac{1}{2} \int (\varepsilon_x)^2 - \frac{1}{3} \int \phi^3 - \int \phi^2 \varepsilon - \int \phi \varepsilon^2 - \frac{1}{3} \int \varepsilon^3$$

Then

$$\overline{F[\varphi]}$$

$\varphi$  is the solitary wave.

$$F[\varphi + \varepsilon] = \left[ \frac{1}{2} \int (\varphi')^2 + \frac{1}{2} \int \varphi^2 - \frac{1}{3} \int \varphi^3 \right] + \int (-\varphi'' + \varphi - \varphi^2) \varepsilon$$

$$+ \underbrace{\frac{1}{2} \int \{ (\varepsilon_x)^2 + \varepsilon^2 - 2\varphi \varepsilon^2 \}}_{\text{quadratic terms}} - \underbrace{\frac{1}{3} \int \varepsilon^3}_{\text{cubic terms}}$$

Remember that  $-\varphi'' + \varphi - \varphi^2 = 0$  (E)

$\Rightarrow L = -\delta_x^2 + 1 - 2\varphi$  is the linearized operator around  $\varphi$

We just showed that

$$\underbrace{F[\varphi + \varepsilon] - F[\varphi]}_{\text{this difference is independent of time}} = \underbrace{F[\varphi + \varepsilon] - \overline{F[\varphi]}}_{\approx} = \frac{1}{2} \underbrace{(L\varepsilon, \varepsilon)}_{\approx} - \frac{1}{3} \int \varepsilon^3 \quad (*)$$

this difference is independent of time

$\approx$   
Kollmann  
(with would like a kind of coercivity for  $L$  is this possible?)

Third step Spectral theory for  $L$  (see for example Reed & Simon vol II)

$\rightarrow D(L) = \{ f \in L^2(\mathbb{R}) : Lf \in L^2(\mathbb{R}) \} = H^2(\mathbb{R}), \quad \underline{L = L^*}$  (Kato-Rellich theorem!).

$L$  is self adjoint

$\rightarrow \ker L = \{ \varphi' \} = \text{ker } \varphi'$

it is trivial to prove that  $\varphi' \in \ker L$ . Indeed, it follows deriving (E).  
 that  $-\varphi''' + \varphi' - 2\varphi\varphi' = 0 \Rightarrow (-\delta_x^2 + 1 - 2\varphi)\varphi' = L\varphi' = 0$   
 To verify that  $\ker L$  is non-degenerated ( $\ker L = \text{ker } \varphi'$ ) we use ODE arguments.  
 Let  $f_1, f_2 \in \ker L$ . Let  $W(f_1, f_2) = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix}$  be the Wronskian of  $\{f_1, f_2\}$ .  
 Then  $\frac{d}{dx} W(f_1, f_2) = \frac{d}{dx} (f_1 f_2' - f_2 f_1') = f_1 f_2'' - f_2 f_1'' = (1 - 2\varphi) f_1 f_2 - (1 - 2\varphi) f_2 f_1 = 0 \Rightarrow f_1 = \lambda f_2$  w.

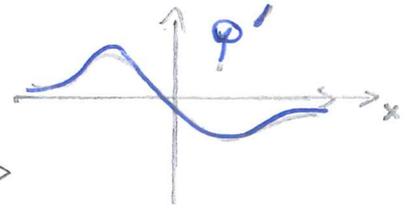
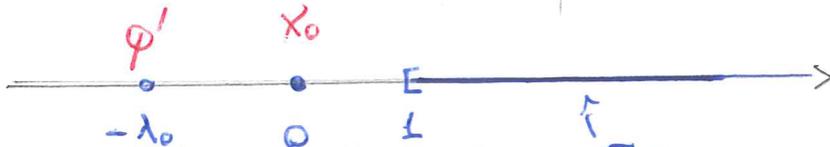
→  $L$  has a unique <sup>negative</sup> eigenvalue  $-\lambda_0 < 0$ . Let  $\begin{cases} X_0 \text{ e.v.m} \\ X_0 > 0, K_0 \in L^2(\Omega) \end{cases}$

$L$  on associated eigen function  $LK_0 = -\lambda_0 K_0$  with  $\|K_0\|_{L^2} = 1$ .

→  $\sigma_{\text{ess}}(L) = [-1, +\infty)$  (by the second part of **Kato-Rolich Theorem**)

The fact that  $-\lambda_0$  is the unique negative eigenvalue comes from **Sturm-Liouville theory**, since  $\varphi'$  has just 1 zero on  $\Omega$

Spectrum of  $L$ .



Fourth step: coercivity properties of  $L$

we deduce the following coercivity properties (from the spectral theorem)

Prop [coercivity properties for  $L$ ]  $\exists \kappa_0, \kappa_1, c_0, c_1, c_2 > 0$  st

①  $(f, \varphi') = (f, X_0) = 0 \Rightarrow (Lf, f) \geq \frac{\kappa_0}{\gamma_0} \|f\|_{L^2}^2$

②  $(f, \varphi') = 0 \Rightarrow (Lf, f) \geq \kappa_0 \|f\|_{L^2}^2 - c_0 |(f, X_0)|^2$

③  $(f, \varphi') = 0 \Rightarrow (Lf, f) \geq \kappa_1 \|f\|_{L^2}^2 - c_1 |(f, \varphi')|^2$

④  $(f, \varphi') = 0 \Rightarrow (Lf, f) \geq \kappa_2 \|f\|_{L^2}^2 - c_2 |(f, \varphi')|^2$

Proof of ① follows from the spectral theorem

$L = L^* \Rightarrow (Lf, f) = \int_{\sigma(L)} \lambda d(E_\lambda f, f) \geq \dots$  (reduction of the identity associated to  $L$ )

②  $(f, \varphi') = 0, (f, X_0) = \gamma_0$

Define  $\tilde{f} := f - \gamma_0 X_0$

Then  $\begin{cases} (\tilde{f}, \varphi') = (f, \varphi') - \gamma_0 (X_0, \varphi') = 0 \\ (\tilde{f}, X_0) = (f, X_0) - \gamma_0 \|X_0\|_{L^2}^2 = 0 \end{cases}$

Thus, we get from ① that

$(L\tilde{f}, \tilde{f}) \geq \kappa_0 \|\tilde{f}\|_{L^2}^2 = \kappa_0 \|f\|_{L^2}^2 - 2\gamma_0 (X_0, f) + \gamma_0^2 \|X_0\|_{L^2}^2$

"  $(Lf, f) - 2\gamma_0 (LX_0, f) + \gamma_0^2 (LX_0, X_0) = (Lf, f) + 2\gamma_0 \lambda_0 (X_0, f) - \lambda_0 \gamma_0^2 \|X_0\|_{L^2}^2$

Then

$$(\lambda f, f) \geq \kappa_0 \|f\|_{H^1}^2 - 2(\kappa_0 + \lambda_0) \underbrace{\gamma_0(\kappa_0, f)}_{\frac{1}{2}(\kappa_0, f)^2} + (\kappa_0 + \lambda_0) \underbrace{\gamma_0^2}_{\frac{1}{2}(\kappa_0, f)^2} = \kappa_0 \|f\|_{H^1}^2 - (\kappa_0 + \lambda_0) \frac{1}{2}(\kappa_0, f)^2 \Rightarrow \textcircled{2}$$

③ Assume now  $(f, \varphi') = (f, \varphi) = 0$  (idea of Weinstein - how we follow an argument by C. Kenig for the proof)

We want to prove that  $(\lambda f, f) \geq \kappa_1 \|f\|_{L^2}^2$ . Then we deduce ③ arguing as in ②

We recall that  $\varphi_c = \phi_c = c \varphi(\sqrt{c}x)$  satisfies  $-\varphi_c'' + c\varphi_c - \varphi_c^2 = 0$  (E)

Let  $\Lambda \varphi := \frac{d}{dc} \varphi_c \Big|_{c=1}$ . Then, we deduce deriving (E) with respect to  $c$  and

taking  $c=1$  that

$$\boxed{\Delta \Lambda \varphi = -\varphi \quad \text{where} \quad \Lambda \varphi = \varphi + \frac{1}{2} x \varphi'}$$

Now we decompose  $\begin{cases} f = \beta_0 \kappa_0 + \tilde{f} & \text{with } (\tilde{f}, \kappa_0) = (\tilde{f}, \varphi') = 0 \\ \Lambda \varphi = \beta_1 \kappa_0 + \tilde{\Lambda} \varphi & \text{with } (\tilde{\Lambda} \varphi, \kappa_0) = (\tilde{\Lambda} \varphi, \varphi') = 0 \end{cases}$

↑ (we remain here that  $(\Lambda \varphi, \varphi') = 0$  by parity)

Then the following properties hold:

(i)  $\boxed{(\lambda f, f) = -\beta_0^2 \lambda_0 + (\lambda \tilde{f}, \tilde{f})} \quad (*)_1$

Indeed, we use that

(ii)  $\boxed{\begin{aligned} (\lambda \Lambda \varphi, \Lambda \varphi) &= -\beta_1^2 \lambda_0 + (\lambda \tilde{\Lambda} \varphi, \tilde{\Lambda} \varphi) = -\rho_0 < 0 \\ \underbrace{-(\varphi, \Lambda \varphi)}_{> 0} &=: -\rho_0 < 0 \end{aligned}} \quad (*)_2$

$$\| \varphi_c \|_{L^2}^2 = c^{3/4} \| \varphi \|_{L^2}^2 \Rightarrow (\varphi, \Lambda \varphi) = \frac{3}{4} \| \varphi \|_{L^2}^2 =: \rho_0 > 0$$

(iii)  $0 = (f, \varphi) = -(f, \lambda \Lambda \varphi) = -(\beta_0 \kappa_0, \lambda(\beta_1 \kappa_0)) - (\tilde{f}, \lambda(\beta_1 \kappa_0)) - (\beta_0 \kappa_0, \lambda(\tilde{\Lambda} \varphi)) - (\tilde{f}, \lambda(\tilde{\Lambda} \varphi))$   
 $\Rightarrow 0 = \lambda_0 \beta_0 \beta_1 \underbrace{\| \kappa_0 \|^2}_{=1} + \lambda_0 \beta_1 \underbrace{(\tilde{f}, \kappa_0)}_0 + \lambda_0 \beta_0 \underbrace{(\kappa_0, \tilde{\Lambda} \varphi)}_0 - (\lambda \tilde{f}, \tilde{\Lambda} \varphi)$   
 $\Rightarrow \boxed{|\lambda_0 \beta_0 \beta_1|^2 = |(\lambda \tilde{f}, \tilde{\Lambda} \varphi)|^2} \quad (*)_3$

We deduce combining  $(*)_1, (*)_2$  and  $(*)_3$  that

$$|(\lambda \tilde{f}, \tilde{\Lambda} \varphi)|^2 = \beta_0^2 \beta_1^2 \lambda_0^2 = [(\lambda \tilde{f}, \tilde{f}) - (\lambda f, f)] [ \rho_0 + (\lambda \tilde{\Lambda} \varphi, \tilde{\Lambda} \varphi) ]$$

Then

$$|(\tilde{f}, \tilde{q})|^2 = \rho_0 (\lambda \tilde{f}, \tilde{f}) - \rho_0 (\lambda f, f) + (\lambda \tilde{f}, \tilde{f}) (\lambda \tilde{q}, \tilde{q}) - (\lambda f, f) (\lambda \tilde{q}, \tilde{q})$$

$$\Rightarrow (\lambda f, f) (\rho_0 + (\lambda \tilde{q}, \tilde{q})) = (\lambda \tilde{f}, \tilde{f}) [\rho_0 + (\lambda \tilde{q}, \tilde{q})] - |(\lambda \tilde{f}, \tilde{q})|^2$$

$$\Rightarrow (\lambda f, f) = \underbrace{(\lambda \tilde{f}, \tilde{f})}_{\geq \kappa_0 \|\tilde{f}\|_Z^2} \left\{ 1 - \frac{|(\lambda \tilde{f}, \tilde{q})|^2}{(\lambda \tilde{f}, \tilde{f}) [\rho_0 + (\lambda \tilde{q}, \tilde{q})]} \right\}$$

by  $\textcircled{1}$  since  $(\tilde{f}, \varphi') = (\tilde{f}, \kappa_0) = 0$

Now observe that  $(z, z) \mapsto (\lambda z, z)$  is a scalar product on  $\{\varphi', \kappa_0\}^\perp$  (it is positive definite).

Then we deduce from the Cauchy-Schwarz inequality that

$$|(\lambda \tilde{f}, \tilde{q})|^2 \leq (\lambda \tilde{q}, \tilde{q}) (\lambda \tilde{f}, \tilde{f})$$

$$\Rightarrow (\lambda f, f) \geq \underbrace{(\lambda \tilde{f}, \tilde{f})}_{\geq \kappa_0 \|\tilde{f}\|_Z^2} \left\{ 1 - \frac{1}{1 + \frac{\rho_0}{(\lambda \tilde{q}, \tilde{q})}} \right\} \geq (1 - \mu_0) \kappa_0 \|\tilde{f}\|_Z^2 \geq 0. \quad (*)$$

$0 < \mu_0 < 1$

Now, coming back to  $(*)$

$$(\lambda f, f) = -\beta_0^2 \lambda_0 + (\lambda \tilde{f}, \tilde{f}) \geq 0 \Rightarrow (\lambda \tilde{f}, \tilde{f}) \geq \beta_0^2 \lambda_0.$$

Then, it follows from  $(*)$  that

$$(\lambda f, f) \geq (1 - \mu_0) (\lambda \tilde{f}, \tilde{f}) \geq \frac{(1 - \mu_0)}{\lambda} (\lambda \tilde{f}, \tilde{f}) + \frac{(1 - \mu_0)}{\lambda} \beta_0^2 \lambda_0 \geq \frac{(1 - \mu_0)}{\lambda} \kappa_0 \|\tilde{f}\|_Z^2 + \frac{(1 - \mu_0)}{\lambda} \beta_0^2 \lambda_0$$

$$\Rightarrow (\lambda f, f) \geq \frac{(1 - \mu_0)}{\lambda} \min\{\kappa_0, \lambda_0\} (\|\tilde{f}\|_Z^2 + \beta_0^2) = \frac{(1 - \mu_0) \min\{\kappa_0, \lambda_0\}}{\lambda} \|\tilde{f}\|_Z^2$$

On the other hand, we compute  $\|f\|_Z^2 = (\tilde{f} + \beta_0 \kappa_0, \tilde{f} + \beta_0 \kappa_0) = \|\tilde{f}\|_Z^2 + \beta_0^2 = \kappa_1$

Sq It is clear that the proof still works by replacing  $\varphi$  by any  $f$  satisfying  $(f, \varphi') = 0$  and  $(\lambda^{-1} f, f) = -\rho_0 < 0$  (when  $f = \varphi$ ,  $\lambda^{-1} \varphi = -\lambda \varphi \Rightarrow (\lambda^{-1} \varphi, \varphi) = -(\lambda \varphi, \varphi) < 0$ )

$$(f, \varphi') = 0 \text{ and } (\lambda^{-1} f, f) = -\rho_0 < 0 \quad \left\{ \begin{array}{l} \text{when } f = \varphi, \lambda^{-1} \varphi = -\lambda \varphi \Rightarrow (\lambda^{-1} \varphi, \varphi) = -(\lambda \varphi, \varphi) < 0 \end{array} \right.$$

④ Once again, it is sufficient to prove that  $(f, \varphi') = (f, \varphi_0) = 0$   
 $\Rightarrow (\lambda f, f) \geq \kappa_2 \|f\|_{H^1}^2$

Let  $0 < \delta < 1$  to be chosen later. We write

$$\begin{aligned}
 (\lambda f, f) &= (1-\delta)(\lambda f, f) + \delta(\lambda f, f) \\
 &\geq (1-\delta)\|f\|_{L^2}^2 + \delta \left( \int (\lambda \varphi)^2 + \int f^2 - 2 \int \varphi f^2 \right) \\
 &\geq (1-\delta)\|f\|_{L^2}^2 + \delta \|f\|_{H^1}^2 - 2\delta \|\varphi\|_{L^\infty} \|f\|_{L^2}^2
 \end{aligned}$$

Fix  $\delta \in (0, 1)$  such that  $2\delta \|\varphi\|_{L^\infty} \leq \frac{1-\delta}{2} \iff \delta(4\|\varphi\|_{L^\infty} + 1) \leq 1$

it works taking  $\delta = \frac{1}{4\|\varphi\|_{L^\infty} + 1}$

It follows that  $(\lambda f, f) \geq \underbrace{\frac{1-\delta}{2}}_{>0} \|f\|_{L^2}^2 + \underbrace{\delta}_{\geq \kappa_2} \|f\|_{H^1}^2$

□

modulation theory

Fifth step: ~~proof of the stability result~~

Recall that  $u(t, x) := \varphi(x - \sigma(t)) + \varepsilon(t, x)$

Then we already computed that

$$F[u_0] - F[\varphi] = F[u_0(t)] - F[\varphi] = \frac{1}{2}(\lambda \varepsilon, \varepsilon) - \frac{1}{3} \int \varepsilon^3$$

Claim:  $\exists C^1$  function  $\sigma: [0, \infty) \rightarrow \mathbb{R}$  such that  $\int \varphi'(x - \sigma(t)) \varepsilon(t, x) dx = 0, \forall t$   
 and  $|\sigma'(t) - 1| \leq C\delta$  (if  $\|u_0 - \varphi\|_{H^1} < \delta$ ). (modulation theory)

Let  $\Psi: \mathbb{R} \times H^1(\mathbb{R}) \rightarrow \mathbb{R}; (\sigma, \varphi) \mapsto \int \varphi'(x - \sigma)(u(x) - \varphi(x - \sigma)) dx$   
 Then  $\Psi$  is smooth,  $\Psi(0, \varphi) = 0$  and

$$\frac{\partial \Psi}{\partial \sigma} \Big|_{(0, \varphi)} = -\int (\varphi'')^2 > 0$$

Thus, it follows from the implicit function theorem that

$\exists \forall$  neighborhood of 0 in  $\mathbb{R}$ ,  $d_1 > 0$  and a  $C^1$  map

$$\sigma: \{u \in H^1(\Omega) : \|u - \varphi\|_{H^1} < d_1\} \rightarrow \mathbb{R} \text{ such that } \Psi(\sigma(u), u) = 0$$

$$\forall u \in H^1(\Omega) \text{ with } \|u - \varphi\|_{H^1} < d_1$$

Then we extend the  $C^1$  map  $\sigma(u)$  to  $U_{d_1} = \{u \in H^1(\Omega) : \inf_{\varphi \in \mathcal{A}} \|u - \varphi\|_{H^1} < d_1\}$

in such a way that  $\sigma(u) = \sigma(u(\cdot + \delta)) + \delta$

we set  $\sigma(\pm) := \sigma(u(\pm, \cdot))$ ,  $\forall \pm \geq 0$ . Then it follows that

$$(*) \quad \int \varphi'(x - \sigma(t)) \varepsilon(t, x) dx = \int \varphi'(x - \sigma(t)) (u(t, x) - \varphi(x - \sigma(t))) dx = 0, \quad \forall t \geq 0.$$

Finally in order to prove the identity  $|\sigma'(t) - 1| \leq c_d$ , we derive (\*) with respect to time.

$$(*)' \Rightarrow \frac{d}{dt} \int \varphi'(x) (u(t, x + \sigma(t)) - \varphi(x)) dx = 0$$

$$\Rightarrow 0 = \frac{d}{dt} \left[ \int \varphi'(x) (u(t, x + \sigma(t)) - \varphi(x)) dx \right]$$

$$= \int \varphi'(x) \left( \underbrace{\partial_t u(t, x + \sigma(t))}_{\substack{\downarrow \\ \text{equation of } u}} + \underbrace{\sigma'(t)}_{\substack{\downarrow \\ \sigma'(t) \int (\varphi')^2 + \dots}} u(t, x + \sigma(t)) \right) dx$$

equation of  $u$

$$\sigma'(t) \int (\varphi')^2 + \dots$$

□

Sixth step: Proof of stability

we remind the ~~the~~ identity

$$\mathcal{F}[u_0] - \mathcal{F}[\varphi] = \frac{1}{2} (L\varepsilon, \varepsilon) - \frac{1}{3} \int \varepsilon^3$$

We fix  $\sigma(t)$  such that  $\int \varphi' \varepsilon(t, \cdot) = 0, \forall t$  by using modulation theory

then we deduce from the coercivity properties of  $L$  that

$$(L\varepsilon, \varepsilon) \geq \kappa \|\varepsilon\|_{H^1}^2 - c |( \varepsilon, \varphi )| \text{ for some constants } \kappa > 0, c > 0.$$

$$\Rightarrow \boxed{\kappa \|\varepsilon\|_{H^1}^2 \leq \underbrace{|\mathcal{F}[u_0] - \mathcal{F}[\varphi]|}_{\leq c \|\mu_0 - \varphi\|_{L^2}^2} + \frac{1}{3} \int |\varepsilon|^3 + c \underbrace{|(\varepsilon, \varphi)|^2}_{\leq c \|\varepsilon\|_{H^1}^3}} \quad (1)$$

$\uparrow$  it remains to control this term  
Sobolev embedding

To control  $|(\varepsilon, \varphi)|^2$  we use the conservation of the mass.

$$\begin{aligned} \mathcal{M}[u_0] &= \mathcal{M}[u](t) = \mathcal{M}[\varphi + \varepsilon](t) = \frac{1}{2} \int \varphi^2 + \int \varphi \varepsilon(t) + \frac{1}{2} \int \varepsilon(t)^2 \\ &\parallel \\ \frac{1}{2} \int \varphi^2 + \int \varphi \varepsilon(0) + \frac{1}{2} \int \varepsilon(0)^2 \end{aligned}$$

$$\Rightarrow \boxed{|(\varphi, \varepsilon)(t)| \leq |(\varphi, \varepsilon)(0)| + \frac{1}{2} \int \varepsilon(0)^2 + \frac{1}{2} \int \varepsilon^2(t)} \quad (2)$$

this estimate is better than the usual Cauchy-Schwarz estimate!

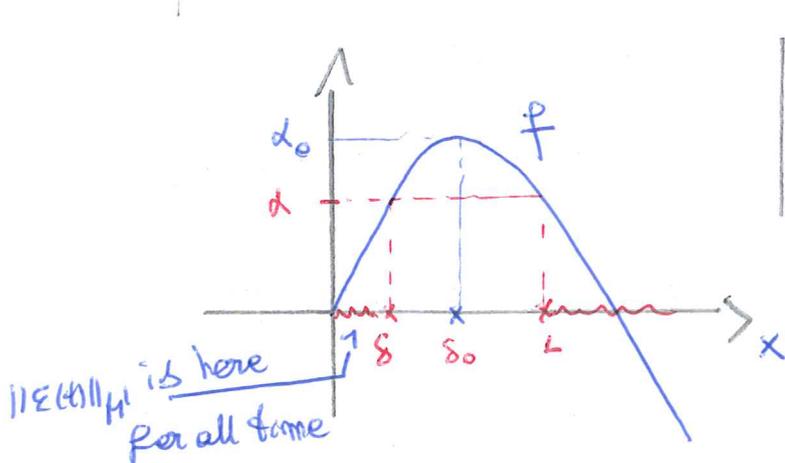
$$|(\varphi, \varepsilon)| \leq \|\varphi\|_{L^2} \|\varepsilon\|_{L^2}$$

Combining (1) and (2), we deduce that

$$(4) \quad K \| \mathcal{E}(t) \|_{H^1}^2 \leq C \| \mathcal{E}(0) \|_{H^1}^2 + C \| \mathcal{E}(t) \|_{H^1}^3 + C \| \mathcal{E}(t) \|_{H^1}^4, \quad \forall t \geq 0.$$

Now we use a ~~calculus~~ <sup>measuring</sup> bootstrap argument based on a ~~calculus~~ <sup>continuity argument</sup> trick:

Let  $f(x) = Kx^2 - C_1x^3 - C_2x^4, x \geq 0$



$$\left\{ \begin{array}{l} \exists \delta > 0, \forall 0 < \delta < \delta_0, \exists \alpha > 0, 0 < \alpha < \alpha_0 \\ f(x) < \alpha \Leftrightarrow 0 \leq x \leq S \text{ or } x > L \end{array} \right. \quad \alpha = \alpha(\delta)$$

Then (4)  $\Leftrightarrow$   $f(\| \mathcal{E}(t) \|_{H^1}) \leq C \| \mathcal{E}(0) \|_{H^1}^2 < \alpha$   
 then (\*)  $\Leftrightarrow$   $f(\| \mathcal{E}(t) \|_{H^1}) \leq C \| \mathcal{E}(0) \|_{H^1}^2 < \alpha$

$$\Rightarrow \left\{ \begin{array}{l} \| \mathcal{E}(t) \|_{H^1} < \delta, \quad \forall t \geq 0. \\ \text{or} \\ \| \mathcal{E}(t) \|_{H^1} \geq L. \end{array} \right.$$

$\leftarrow$  This option is excluded by continuity ( $\mathcal{E} \in C^0([0, \infty); H^1(\mathbb{R}))$ ) if  $\| \mathcal{E}(0) \|_{H^1}$  is chosen small enough

□

# I - Asymptotic stability of solitary waves

Theo [Floer, Merle & Luo, 2005, 2008]

The solitary waves of KdV are asymptotically stable in the energy space, i.e.  
Let  $c_0 > 0, \forall \beta > 0, \exists \epsilon_0 > 0, K_0 > 0$  s.t. if  $\|u_0 - \varphi_{c_0}\|_{H^1} < \epsilon \leq \epsilon_0$ , then:  
 $\exists c_+ > 0$  with  $|c_+ - c_0| < K_0 \epsilon$ ,  $\exists \sigma \in C^1(\mathbb{R} \rightarrow \mathbb{R})$  such that  
$$\begin{cases} u(x,t) - \varphi_{c_+}(\cdot - \sigma(t)) \longrightarrow 0 \text{ in } H^1(x > \beta t) \\ c'(t) \longrightarrow c_+ \end{cases}$$

Eq ① First proof by Pego & Weinstein with strong decay assumptions on the initial data (1984)

② The convergence cannot be in the whole space  $H^1(\mathbb{R})$

Indeed if it was, one would have

$$(*) \begin{cases} E[u_0] = E[u](t) = E[\varphi] \\ M[u_0] = M[u](t) = M[\varphi] \end{cases}$$

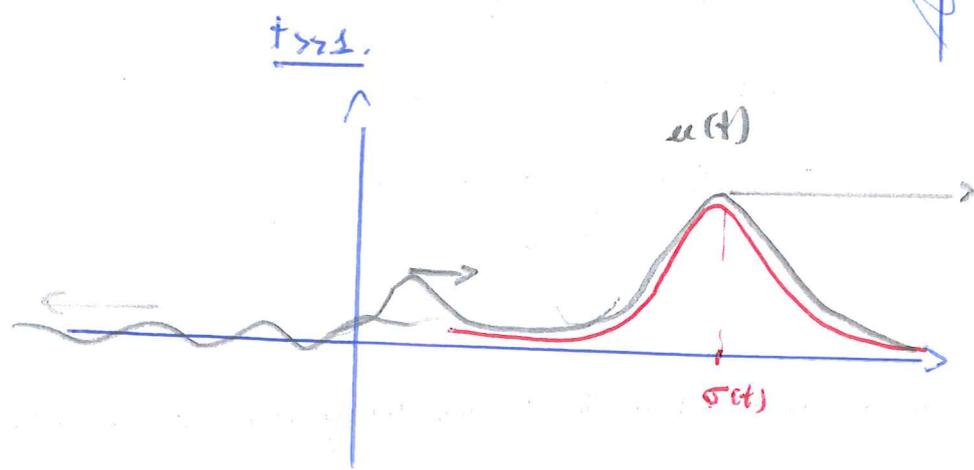
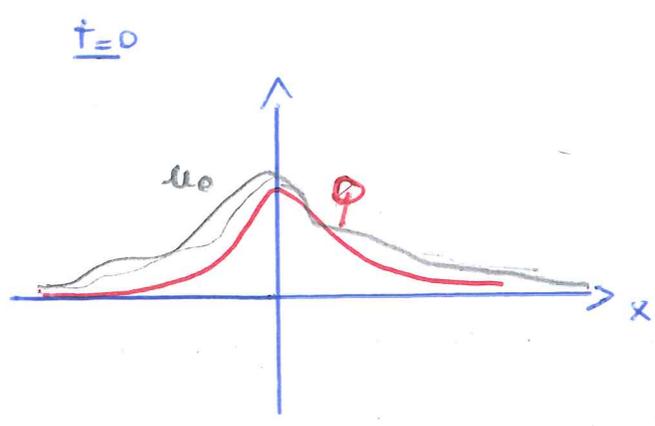
on the other hand, one has  
Variational characterization of  $\varphi$

$\varphi_c$  is the unique (up to translation) minimizer of  $E$  with fixed  $L^2$ -norm, i.e.  $\inf_{\|u\|_{L^2} = \lambda} E[u]$  is attained in  $\varphi_c$  where  $\|\varphi_c\|_{L^2} = \lambda$ .

(This can be proved by using the concentration compactness method  
see Cazenave-Lions (1982) and provides another proof of the orbital stability result

Then by (\*) we would deduce that  $u_0 \equiv \varphi$ .

Physically: the dispersion is moving to the left  
+ smaller solitons can emerge on the left.



Strategy of the proof:

(i) Fix  $c_0 = 1$

(ii) we have  $\|u_0 - \varphi\|_{H^1} < \epsilon$  by hypothesis

$\implies \sup_{t \in \mathbb{R}} \|u(x, \cdot + \sigma(t)) - \varphi\|_{H^1} \leq K_0 \epsilon$  (for some  $C^1$  function  $\sigma$ )  
*(orbital stability result)*

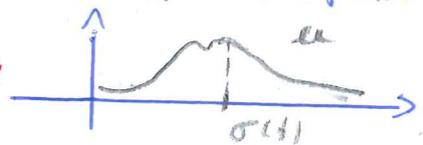
(iii) Then  $\exists t_0, \tau_0, \tilde{u}_0 \in H^1(\mathbb{R})$  st  $u(t_0, \cdot + \sigma(t_0)) \xrightarrow{H^1} \tilde{u}_0$  (weak  $C^1$ )

(iv) let  $\tilde{u}$  be the solution evolving from  $\tilde{u}_0$ . Then we prove that

(1)  $\forall A > 0, u(t_n, \cdot + \sigma(t_n)) \longrightarrow \tilde{u}_0$  in  $H^1(x_1, x_2 - A)$

(2)  $\begin{cases} \| \tilde{u}(t, \cdot + \tilde{\sigma}(t)) - \varphi \|_{H^1} \leq K_0 \epsilon \\ | \tilde{u}(t, x + \tilde{\sigma}(t)) | \leq \tilde{k} e^{-\delta |x|}, \forall t, x \in \mathbb{R} \end{cases}$  (for some  $\tilde{k} > 0, \delta > 0$ )

$\tilde{u}$  is localized like a soliton wave

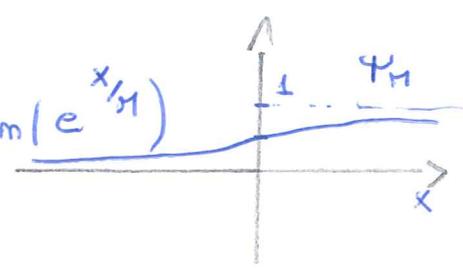


Rq ① + ② are proved by using monotonicity arguments

Monotonicity for KdV

Let  $u$  be a solution of KdV satisfying (\*).

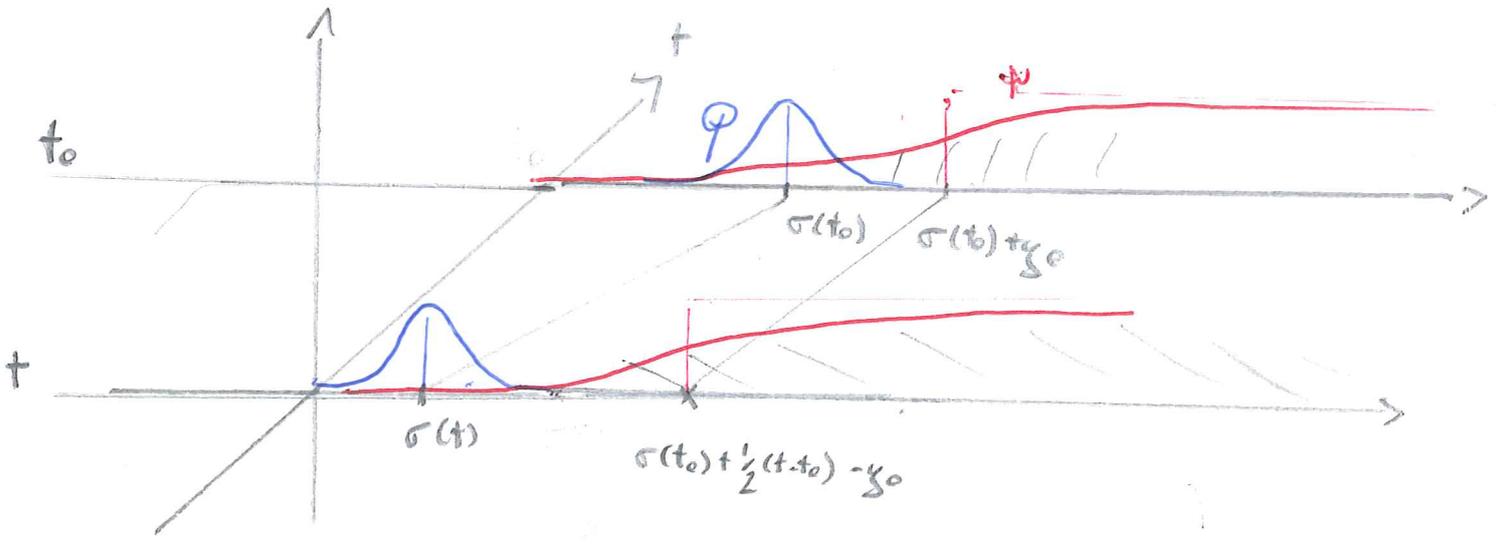
Let  $\tilde{x} = x - \sigma(t_0) + \frac{1}{2}(t_0 - t) - y_0$ ,  $\psi_{\mu}(x) = \frac{2}{\sigma} \arctan(e^{x/\mu})$



and  $I_{y_0, t_0}(t) = \int u^3(t, x) \psi_{\mu}(\tilde{x}) dx$

Then  $I_{y_0, t_0}(t_0) - I_{y_0, t_0}(t) \leq e^{-y_0/\mu}$ ,  $\forall y_0 > 0, \forall t_0 \in \mathbb{R}, \forall t \leq t_0$

(the proof relies on Kato's ideas)



t = t\_0  $\tilde{x} = 0 \Leftrightarrow x = \sigma(t_0) + y_0$

t  $\tilde{x} = 0 \Leftrightarrow x = \sigma(t_0) + \frac{1}{2}(t_0 - t) - y_0$

① + ②

In resonance  $\tilde{u}$  is localized like a solitary wave and close to a solitary wave

⑤ Rigidity theorem  $\Rightarrow \tilde{u} \equiv \varphi_{c+}$

# Rigidity Theorem [Liouville property for KdV]

## Theorem [Mortel-Rode 2000]

Let  $u \in C^0(\mathbb{R}; H^1(\mathbb{R}))$  be a solution of KdV such that

$$\textcircled{1} \|u(t, \cdot + \sigma(t)) - \varphi\|_{H^1} \leq \varepsilon, \forall t \in \mathbb{R}$$

$$\textcircled{2} |u(t, \cdot + \sigma(t))| \lesssim e^{-\delta|x|} \quad \forall x, t \in \mathbb{R}$$

for some  $C^1$  function  $\sigma \in C^1$ .

Then  $\exists \left\{ \begin{array}{l} c_+ \text{ close to } \frac{1}{2} \\ x_0 \in \mathbb{R} \end{array} \right.$  such that  $u(t, x) = \varphi_{c_+}(x - c_+t - x_0)$

## Theorem [Mortel 2006 - Liouville linear]

~~Let  $u$  be a solution.~~

Let  $u \in C^0(\mathbb{R}; H^1(\mathbb{R}))$  be a solution of  $\left\{ \begin{array}{l} \delta_t u = \delta_x^2 u \\ \mathcal{L} = -\delta_x^2 + 1 - 2\varphi \end{array} \right.$

such that  $|u(t, x)| \lesssim e^{-\delta|x|}, \forall x, t \in \mathbb{R}$

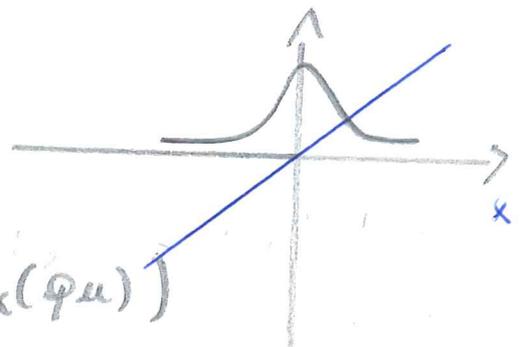
Then  $u(t, x) = a\varphi'$  for  $a \in \mathbb{R}$ .

## Proof [Mortel 2006]

The proof is based on a **virial identity** (understand what happens around  $\varphi$ ).

First try: direct virial

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int x u^2 &= \int x u \delta_t u = \int x u \delta_x^2 u \\ &= \int x u (-\delta_x^3 u + \delta_x u - 2\delta_x(\varphi u)) \end{aligned}$$



$$\begin{aligned} \Rightarrow \frac{1}{2} \frac{d}{dt} \int x u^2 &= - \int x u \delta_x^3 u + \int x u \delta_x u - 2 \int x u \delta_x (\varphi u) \\ &= + \int u \delta_x^2 u + \int x \delta_x u \delta_x^2 u - \frac{1}{2} \int u^2 - 2 \int x \varphi' u^2 - 2 \int x \varphi u \delta_x u \\ &= - \frac{3}{2} \int (\delta_x u)^2 - \frac{1}{2} \int u^2 + \int \varphi u^2 - \int x \varphi' u^2 \end{aligned}$$

$$\begin{aligned} \Rightarrow \frac{1}{2} \frac{d}{dt} \int x u^2 &= - \int (\delta_x u)^2 - \frac{1}{2} \int \{ (\delta_x u)^2 + u^2 - 2\varphi \} - \int x \varphi' u^2 \\ &= - \underbrace{\int (\delta_x u)^2}_{\leq 0} - \frac{1}{2} \underbrace{(2u, u)}_{\leq} - \underbrace{\int x \varphi' u^2}_{\leq 0} \end{aligned}$$

with good orthogonality conditions  
(coercivity of  $L$ )

$$-\lambda \|u\|_{H^1}^2$$

$\geq 0$

wrong sign.

Second try: consider the dual problem

$$\sigma = Lu - d_0 \varphi, \quad d_0 \in \mathcal{D}$$

$$\text{Then } \partial_t \sigma = L \partial_t u = L \delta_x \lambda u = L \delta_x \sigma + d_0 \lambda \varphi'$$

$$\textcircled{1} (\sigma, \varphi') = (Lu, \varphi') - d_0 (\varphi, \varphi') = (u, \lambda \varphi') = 0$$

$$\textcircled{2} \frac{d}{dt} (\sigma, \lambda \varphi) = (\partial_t \sigma, \lambda \varphi) = (L \delta_x \sigma, \lambda \varphi) = (\delta_x \sigma, \lambda \lambda \varphi) = (\sigma, \varphi') \Big|_{t=0}$$

if we choose  $d_0$  st  $(\sigma|_{t=0}, \lambda \varphi) = 0$

$$\Rightarrow (\sigma, \lambda \varphi) = 0, \quad \forall t.$$

we compute

$$\frac{1}{2} \frac{d}{dt} \int x \sigma^2 = \int x \sigma \partial_t \sigma = \int x \sigma L \delta_x \sigma = - \int x \sigma \delta_x^3 \sigma + \int x \sigma \delta_x \sigma - 2 \int x \sigma \varphi \delta_x \sigma$$

Following the same computations as for  $w$ :

$$\frac{1}{2} \frac{d}{dt} \int x \sigma^2 = -3 \frac{1}{2} \int (\partial_x \sigma)^2 - \frac{1}{2} \int \sigma^2 - \underbrace{\int x \varphi \partial_x (\sigma^2)}_{=0} + \int \varphi \sigma^2 + \int x \varphi' \sigma^2$$

$$\Rightarrow \boxed{\frac{1}{2} \frac{d}{dt} \int x \sigma^2 = - \underbrace{\int (\partial_x \sigma)^2}_{\leq 0} - \frac{1}{2} \underbrace{\int \sigma^2}_{\leq -k \|\sigma\|_{H^1}^2} + \underbrace{\int x \varphi' \sigma^2}_{\leq 0}}$$

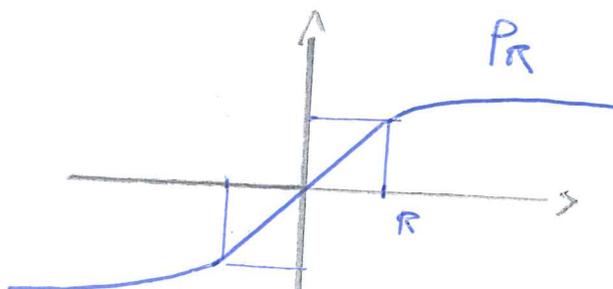
coercivity of  $\sigma$  with respect to  $(\Delta \varphi, \varphi')$

↳ this term this term has the good sign

we conclude that

$$\boxed{\frac{d}{dt} \int x \sigma^2 \leq -k \int \sigma^2}$$

totally we work with  $\varphi_R \stackrel{\Delta}{=} t$



we have semilocally

$$\boxed{\frac{d}{dt} \int \varphi_R \sigma^2 \leq -k \int \varphi_R' \sigma^2} \quad (*)$$

Integrating (\*) between  $\tau$  and  $s \Rightarrow \int_{\tau}^s \int \varphi_R' \sigma^2 \leq \int \varphi_R \sigma^2(s) - \int \varphi_R \sigma^2(\tau)$

$$\leq C \|\sigma\|_{L^2}^2 + \frac{1}{k}$$

( $L^2$ -compactness property)

$$\Rightarrow \exists \begin{cases} \tau_n \nearrow \infty \\ s_n \nearrow \infty \end{cases} \stackrel{\Delta}{=} t$$

$$\boxed{\begin{cases} \int \varphi_R' \sigma^2(\tau_n, x) dx \rightarrow 0 \\ \int \varphi_R' \sigma^2(-s_n, x) dx \rightarrow 0 \end{cases} \quad n \rightarrow \infty}$$

$$(*) \Rightarrow \begin{cases} \int \sigma^2(\tau_n, x) dx \rightarrow 0 \\ \int \sigma^2(-s_n, x) dx \rightarrow 0 \end{cases}$$

Integrating (\*) between  $-S_n$  and  $T_m$  and using (4)

$$\Rightarrow \int_{-N}^{+N} \int_{\mathbb{R}} p'_R \sigma^2 dx dt = 0 \Rightarrow \sigma \equiv 0.$$

Best  $\sigma = Lu - \alpha_0 \varphi = 0 \Leftrightarrow Lu = \alpha_0 \varphi$

$$\Rightarrow \boxed{u = \beta(t) \varphi' + \gamma(t) \Lambda \varphi}$$

$$\begin{cases} \Lambda \varphi = -\varphi \\ \Lambda \varphi \perp \varphi' \end{cases}$$

Now we compute  $\partial_t u = \beta' \varphi' + \gamma' \Lambda \varphi$

$$\partial_x Lu = \beta \partial_x \Lambda \varphi + \gamma \partial_x \Lambda \varphi = -\gamma \varphi'$$

$$\Rightarrow \begin{cases} \beta' = -\gamma \\ \gamma' = 0 \end{cases} \Rightarrow \begin{cases} \gamma \equiv \gamma_0 \\ \beta(t) = -\gamma_0 t + \beta_0 \end{cases}$$

$$\Rightarrow u = (-\gamma_0 t + \beta_0) \varphi' + \gamma_0 \Lambda \varphi \} \Rightarrow \gamma_0 \equiv 0$$

$u$  bounded (in time)

$$\Downarrow \boxed{u = \beta_0 \varphi'}$$

