Simplicity and the unique trace property for some L^p-operator algebras

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L^2-operator algebra is not necessarily self-adjoint!

Let G be discrete.

$$\lambda\colon G \to \mathcal{B}(\ell^p(G)), \quad g \mapsto \lambda_g(\colon \delta_h \mapsto \delta_{gh})$$
$$\lambda\colon \mathbb{C}G \to \mathcal{B}(\ell^p(G))$$
$$\mathrm{F}^p_{\mathrm{r}}(G) = \overline{\lambda(\mathbb{C}G)} \subset \mathcal{B}(\ell^p(G)).$$

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The canonical trace $\tau\colon \mathrm{F}^p_{\mathrm{r}}(G) \to \mathbb{C}$ is a unital trace satisfying

$$\tau(a) = \gamma_e$$

for $a = \sum_{\text{finite}} \gamma_g \lambda_g \in \lambda(\mathbb{C}G)$

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- The class of Banach algebras that can be isometrically on an L^p-space, for $p \neq 2$ is not closed under quotients. Example, there is a quotient algebra of $\mathrm{F}^p(\mathbb{Z})$ that has no isometric representation on any L^p-space. (Gardella- Thiel 2016)

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- Let $p \in [1, \infty) \setminus \{2\}$, G discrete group, and assume that $\iota \colon \mathrm{F}^p(G) \to \mathrm{F}^p_{\mathrm{r}}(G)$ is an isometric isomorphism. Is G amenable as in the C^*-setting?

On the similarities ...

Powers, 1975
$\mathrm{C}^*_{\mathrm{r}}(\mathbb{F}_2)$ is simple and has a unique trace.

Hejazian-P., 2015
$\mathrm{F}^p_{\mathrm{r}}(\mathbb{F}_2)$ for $p \in (1, \infty)$ is simple and has a unique trace.

Key tool

Lemma
Let $p \in (1, \infty)$. *Let* $a \in \mathrm{F}^p_{\mathrm{r}}(\mathbb{F}_2)$, *let* $\epsilon > 0$. *Then there exist* $k \in \mathbb{N}$ *and* $h_1, \cdots, h_k \in \mathbb{F}_2$ *such that the averaging operator*

$$T(b) = \frac{1}{k} \sum_{j=1}^{k} u_{h_j} \, b \, u_{h_j}^{-1}$$

satisfies $\|T(a) - \tau(a) \cdot 1\| \leq \epsilon$.

Theorem
$\mathrm{F}^p_{\mathrm{r}}(\mathbb{F}_2)$ for $p \in (1, \infty)$ is simple.

Proof.

- Let $\mathcal{I}$ be a nonzero ideal in $\mathrm{F}^p_{\mathrm{r}}(\mathbb{F}_2)$.
- There exists $a \in \mathcal{I}$ such that $\tau(a) = 1$
- Apply the averaging operator to $a - \tau(a) \cdot 1$ and $\epsilon = \frac{1}{2}$
- so there exist $k \in \mathbb{N}$ and $h_1, \cdots, h_k \in \mathbb{F}_2$ such that

$$\|\frac{1}{k}\sum_{j=1}^{k} u_{h_j}\, a\, u_{h_j}^{-1} - 1\| \leq \frac{1}{2}.$$

- $\mathcal{I}$ contains an invertible element. Therefore $\mathcal{I} = \mathrm{F}^p_{\mathrm{r}}(\mathbb{F}_2)$.

□

Theorem
$\mathrm{F}^p_{\mathrm{r}}(\mathbb{F}_2)$ *for* $p \in (1, \infty)$ *has a unique trace.*

Proof.

- Let σ be a unital trace on $\mathrm{F}^p_{\mathrm{r}}(\mathbb{F}_2)$, $a \in \mathrm{F}^p_{\mathrm{r}}(\mathbb{F}_2)$, and $\epsilon > 0$.
- By the key lemma we have the averaging operator T such that $\|\frac{1}{k}\sum_{j=1}^{k} u_{h_j} a u_{h_j}^{-1} - \tau(a) \cdot 1\| \leq \epsilon$.
- Let $c := \frac{1}{k}\sum_{j=1}^{k} u_{h_j} a u_{h_j}^{-1}$
- $\sigma(c) = \sigma(a), \quad \sigma(\tau(a) \cdot 1) = \tau(a)$
- $|\tau(a) - \sigma(a)| = |\sigma(c - \tau(a) \cdot 1)| \leq \epsilon$
- $\sigma(a) = \tau(a)$.

$\square$

What about $p = 1$?

$\mathrm{F}^1_{\mathrm{r}}(G)$ is not simple, for G countable and discrete.

$$\ell^1(G) \cong \mathrm{F}^1_{\mathrm{r}}(G)$$

- trivial homomorphism $\varphi \colon G \to \mathbb{C}$
- induced to $\tilde{\varphi} \colon \ell^1(G) \to \mathbb{C}$
- $\ker(\tilde{\varphi})$ is a non trivial ideal.

more generally ...

Theorem : Hejazian-P., 2015
G Powers groups, A simple, G-L^p algebra, $A \rtimes_{\mathrm{r},p} G$ is simple. If A has a unique trace, the crossed product has a unique trace as well.

Thank you!