Quantum Harmonic Analysis and its Applications

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Harmonic Analysis

Harmonic analysis is a branch of mathematics concerned with the representation of functions or signals as the superposition of basic waves, and the study of and generalization of the notions of Fourier series and Fourier transforms (i.e. an extended form of Fourier analysis).

Quantum mechanics

Quantum mechanics describes the physics of atoms and molecules. His mathematical formulation is linked with the representation theory of the Heisenberg group and the theory of states and density matrices.

Harmonic analysis in a nutshell

Basic notions and facts

- Translation $T_x f(t) = f(t - x)$
- Convolution $(f * g)(x) = \int f(y)g(x - y)dy = \int f(y)T_y\check{g}(x)\, dy$
- Fourier transform $\widehat{f}(\omega) = \mathcal{F}f(\omega) = \int f(t)e^{-2\pi i\omega t}dt$
- $\mathcal{F}(f * g) = \mathcal{F}(f)\mathcal{F}(g)$

Algebraic structure

- $L^1(\mathbb{R}^d)$ is a Banach convolution algebra: $\|f * g\|_1 \leq \|f\|_1\|g\|_1$.
- For $1 \leq p < \infty$ the space $L^p(\mathbb{R}^d)$ is a Banach module over $L^1(\mathbb{R}^d)$ for the module action $(f, g) \mapsto f * g$ for $f \in L^p(\mathbb{R}^d)$ and $f \in L^1(\mathbb{R}^d)$.

Riemann-Lebesgue

The Fourier transform of an $L^1(\mathbb{R}^d)$ function vanishes at ∞.

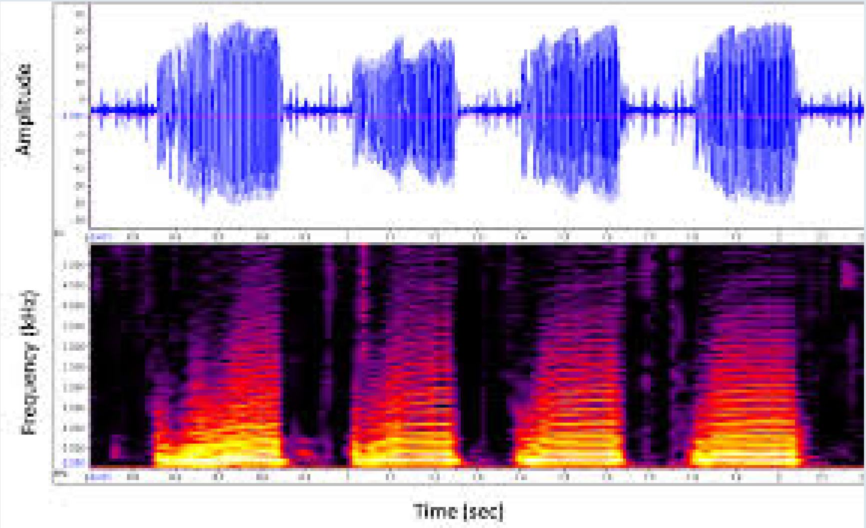
Hausdorff-Young

For $p \in [1, 2]$ and $q = p/(p-1)$ we have $\|\widehat{f}\|_q \leq p^{1/2p} q^{-1/2q} \|f\|_p$.

Wiener Tauberian Theorem

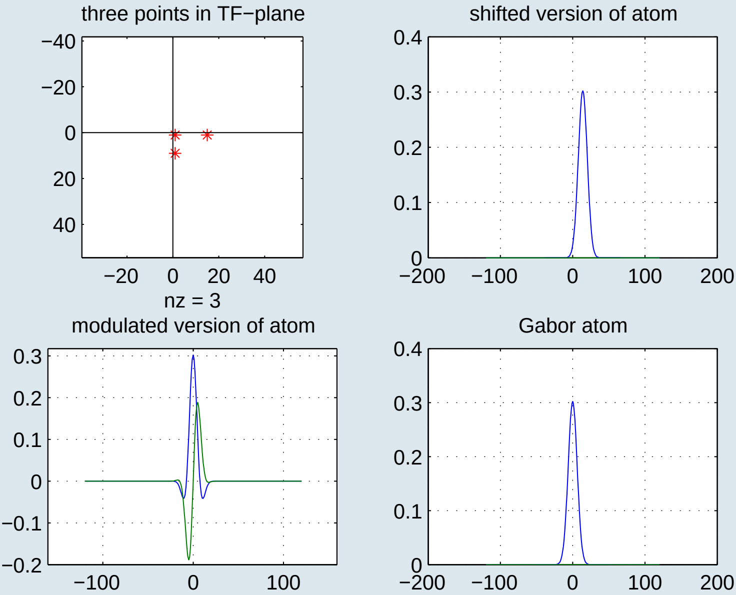
- $\mathrm{span}\{T_z f : z \in \mathbb{R}\}$ is norm dense in $L^1(\mathbb{R})$ if and only if $\{z \in \mathbb{R} : \widehat{f}(z) = 0\}$ is empty.
- $\mathrm{span}\{T_z f : z \in \mathbb{R}\}$ is norm dense in $L^2(\mathbb{R})$ if and only if $\{z \in \mathbb{R} : \widehat{f}(z) = 0\}$ has Lebesgue measure zero.
- $\mathrm{span}\{T_z f : z \in \mathbb{R}\}$ is a weak* dense subspace of $L^\infty(\mathbb{R})$ if and only if $\{z \in \mathbb{R} : \widehat{f}(z) = 0\}$ has dense complement.

Spectrogram



Blurred image

Basic building blocks



Basic notions – Time-frequency analysis

Given $\phi, \psi : \mathbb{R}^d \to \mathbb{C}$ and $z = (x, \omega) \in \mathbb{R}^{2d}$.

- *translation operator* T_x by $T_x\psi(t) = \psi(t - x)$,
- *modulation operator* M_ω by $M_\omega\psi(t) = e^{2\pi i\omega\cdot t}\psi(t)$
- *time-frequency shifts* $\pi(z)$ by $\pi(z) = M_\omega T_x$
- *symmetric time-frequency shifts* $M_{\omega/2}T_xM_{\omega/2} = e^{-\pi ix\cdot\omega}\pi(z)$

Commutation relation

$$T_xM_\omega = e^{2\pi ix\omega}M_\omega T_x$$

$$\begin{aligned} T_xM_\omega\psi(t) &= e^{2\pi i(t-x)}\psi(t-x) \\ &= e^{-2\pi ix\omega}e^{2\pi i(t-x)}\psi(t-x) \\ &= e^{-2\pi ix\omega}M_\omega T_x\psi(t) \end{aligned}$$

Basic notions – Time-frequency analysis

Time-frequency representations

- *short-time Fourier transform* (STFT)
  $V_\phi\psi(z) = \langle \psi, \pi(z)\phi \rangle = \int_{\mathbb{R}^d} \psi(t)\overline{\phi(t-x)}e^{-2\pi i\omega t}dt$
- *ambiguity function* $A(\phi,\psi)(z) = \langle \psi, \pi(z)\phi \rangle = e^{\pi i x\omega} V_\phi\psi(z)$
- *spectrogram* $|V_\phi\psi|^2$
- *cross-Wigner distribution*
  $W(\psi,\phi)(x,\omega) = \int_{\mathbb{R}^d} \psi\left(x+\frac{t}{2}\right)\overline{\phi\left(x-\frac{t}{2}\right)}e^{-2\pi i\omega\cdot t}\, dt.$

Basic notions – Time-frequency analysis

Symplectic Fourier transform

Standard symplectic form σ is defined for $(x_1, \omega_1), (x_2, \omega_2) \in \mathbb{R}^{2d}$ by $\sigma(x_1, \omega_1; x_2, \omega_2) = \omega_1 \cdot x_2 - \omega_2 \cdot x_1$ and the *symplectic Fourier transform* $\mathcal{F}_\sigma f$ of f by

$$\mathcal{F}_\sigma f(z) = \iint_{\mathbb{R}^{2d}} f(z') e^{-2\pi i \sigma(z,z')} \, dz'$$

- The symplectic Fourier transform and the regular Fourier transform $\mathcal{F} f(z) = \iint_{\mathbb{R}^{2d}} f(z') e^{-2\pi i z \cdot z'} \, dz'$ are related by $\mathcal{F}_\sigma f(x, \omega) = \mathcal{F} f(\omega, -x)$.
- $\mathcal{F}_\sigma^2 = I$ and $\mathcal{F}_\sigma^{-1} = \mathcal{F}_\sigma$
- $W(\psi, \phi) = \mathcal{F}_\sigma A(\psi, \phi)$

Shift for operators

One can obtain a unitary representation α of $\mathbb{R}^{2d}$ on the Hilbert-Schmidt operators $\mathcal{T}^2$ by defining

$$\alpha_z(A) = \pi(z)A\pi(z)^* \qquad \text{for } \ z \in \mathbb{R}^{2d}, \ \ A \in B(L^2(\mathbb{R}^d)).$$

$\alpha_z \alpha_{z'} = \alpha_{z+z'}$, and thus α is a shift or translation of operators.

Remark

Since we defined α by $\alpha_z T = \pi(z)T\pi(z)^*$ for $z = (x, \omega) \in \mathbb{R}^{2d}$, we can modify π by any phase factor without affecting α. In particular the family of representations $\pi_\lambda(z) = T_{\lambda x} M_\omega T_{(1-\lambda)x}$ would all give the same α for $\lambda \in [0, 1]$.

Observations

- The map $z \mapsto \alpha_z T$ is continuous from $\mathbb{R}^{2d}$ to $K(L^2(\mathbb{R}^d))$ for any fixed $T \in K(L^2(\mathbb{R}^d))$.
- The map $z \mapsto \alpha_z A$ is weak*-continuous from $\mathbb{R}^{2d}$ to $B(L^2(\mathbb{R}^d))$ for any fixed $A \in B(L^2(\mathbb{R}^d))$.

Schrödinger representation

Time-frequency shifts – Basic identities

- $\pi(z)^* = e^{2\pi i x\omega}\pi(-z)$
- $\pi(z+z') = e^{2\pi i x\omega'}\pi(z)\pi(z')$
- $\pi(z)\pi(z') = e^{2\pi i\sigma(z,z')}\pi(z')\pi(z)$, i.e. $\alpha_z\pi(z') = e^{2\pi i\sigma(z,z')}\pi(z')$

Projective representation

In other words, the time-frequency shifts $\pi(z)$ give a projective representation of $\mathbb{R}^{2d}$ on $L^2(\mathbb{R}^d)$ with respect to the cocycle $c(z,z') = e^{-2\pi i\omega'\cdot x}$.

Moyal's identity

If $\psi_1, \psi_2, \phi_1, \phi_2 \in L^2(\mathbb{R}^d)$, then $V_{\phi_i}\psi_j \in L^2(\mathbb{R}^{2d})$ for $i, j \in \{1, 2\}$, and the relation

$$\langle V_{\phi_1}\psi_1, V_{\phi_2}\psi_2\rangle = \langle \psi_1, \psi_2\rangle\overline{\langle \phi_1, \phi_2\rangle}$$

holds, where the leftmost inner product is in $L^2(\mathbb{R}^{2d})$ and those on the right are in $L^2(\mathbb{R}^d)$.

Moyal's identity implies a reconstruction formula:

Reconstruction formula

Suppose $\phi, \psi \in L^2(\mathbb{R}^d)$. Then for any $\xi \in L^2(\mathbb{R}^d)$ we have

$$\xi = \langle \psi, \phi\rangle^{-1} \iint_{\mathbb{R}^d} V_\phi\xi(z)\pi(z)\psi\, dz.$$

Consequence

The Schrödinger representation is irreducible.

Spectral decomposition of compact operators

Let S be a compact operator on $L^2(\mathbb{R}^d)$. There exist two orthonormal sets $\{\psi_n\}_{n\in\mathbb{N}}$ and $\{\phi_n\}_{n\in\mathbb{N}}$ in $L^2(\mathbb{R}^d)$ and a sequence $\{s_n(S)\}_{n\in\mathbb{N}}$ of positive numbers with $s_n(S) \to 0$, such that S may be expressed as

$$S = \sum_{n\in\mathbb{N}} s_n(S)\psi_n \otimes \phi_n,$$

with convergence in the operator norm. The numbers $\{s_n(S)\}_{n\in\mathbb{N}}$ are called the *singular values* of S, and are the eigenvalues of the operator $|S| = (S^*S)^{1/2}$.

Schatten classes

For $1 \leq p < \infty$ we define the *Schatten class* $\mathcal{T}^p$ of operators by

$$\mathcal{T}^p = \{T \text{ compact} : (s_n(T))_{n\in\mathbb{N}} \in \ell^p\}.$$

We will also write $\mathcal{T}^\infty = B(L^2(\mathbb{R}^d))$ with $\|\cdot\|_{\mathcal{T}^\infty}$ given by the operator norm to simplify the statement of some results.

Basic properties

The Schatten class $\mathcal{T}^p$ becomes a Banach space under pointwise addition and scalar multiplication in the norm

$$\|S\|_{\mathcal{T}^p} = \left(\sum_{n\in\mathbb{N}} s_n(S)^p\right)^{1/p}.$$

We have $\|\cdot\|_{B(L^2(\mathbb{R}^d))} \leq \|\cdot\|_p \leq \|\cdot\|_1$ for $1 \leq p \leq \infty$. Furthermore, the spaces $\mathcal{T}^p$ are ideals in $B(L^2(\mathbb{R}^d))$, i.e. for $A \in B(L^2(\mathbb{R}^d))$ and $T \in \mathcal{T}^p$ we have that $AT, TA \in \mathcal{T}^p$

Recall that an operator $S \in B(L^2(\mathbb{R}^d))$ is *positive* if $\langle S\psi, \psi \rangle \geq 0$ for any $\psi \in L^2(\mathbb{R}^d)$.

Trace class operators

For a positive operator $S \in B(L^2(\mathbb{R}^d))$, the *trace* of S is defined to be

$$\mathrm{tr}(S) = \sum_{n\in\mathbb{N}} \langle S e_n, e_n \rangle, \tag{1}$$

where $\{e_n\}_{n\in\mathbb{N}}$ is an orthonormal basis for $L^2(\mathbb{R}^d)$.
If $S \in \mathcal{T}^1$, then $\mathrm{tr}(S)$ is well-defined and a simple calculation shows that

$$\mathrm{tr}(S) = \sum_{n\in\mathbb{N}} s_n(S),$$

where the sum of singular values converges by the definition of $\mathcal{T}^1$. For this reason the class $\mathcal{T}^1$ is often referred to as *trace class operators*.

Remark

This definition is independent of the orthonormal basis used, and the trace is linear and satisfies $\mathrm{tr}(ST) = \mathrm{tr}(TS)$. However, the expression in the defintion may well be infinite, and is not well-defined for a general non-positive operator S.

Duality

Let $1 \leq p < \infty$, and let q be the number determined by $\frac{1}{p} + \frac{1}{q} = 1$. The dual space of $\mathcal{T}^p$ is $\mathcal{T}^q$, and the duality may be given by

$$\langle T, S \rangle_{\mathcal{T}^q, \mathcal{T}^p} = \mathrm{tr}(TS)$$

for $S \in \mathcal{T}^p$ and $T \in \mathcal{T}^q$.

Hilbert-Schmidt operators

Another well-known Schatten class is $\mathcal{T}^2$, known as the *Hilbert-Schmidt operators*. $\mathcal{T}^2$ is a Hilbert space under the inner product $\langle S, T \rangle_{\mathcal{T}^2} := \mathrm{tr}(ST^*)$ for $S, T \in \mathcal{T}^2$.

Remark

The Schatten classes behave analogously to the L^p-spaces of functions – the duality relations are the same, and both $L^1(\mathbb{R}^{2d})$ and $\mathcal{T}^1$ have a natural linear functional given by the integral and trace, respectively. The intuition that L^p corresponds to $\mathcal{T}^p$ will often be useful.

Trace-Properties

Let $S \in \mathcal{T}^1(\mathcal{H})$, $A \in B(\mathcal{H})$.

1. $S^* \in \mathcal{T}^1(\mathcal{H})$, and $\mathrm{tr}(S^*) = \overline{\mathrm{tr}(S)}$.
2. $\mathrm{tr}(AS) = \mathrm{tr}(SA)$.
3. $\sum_{n\in\mathbb{N}} |\langle ASe_n, e_n\rangle| \leq \|A\|_{B(L^2)}\|S\|_{\mathcal{T}^1}$.
4. $|\mathrm{tr}(AS)| \leq \|A\|_{B(L^2)}\|S\|_{\mathcal{T}^1}$.

Parity operator

we define the analogue of $f \mapsto \check{f}$ for an operator $A \in B(L^2(\mathbb{R}^d))$ by

$$\check{A} = PAP,$$

where P is the parity operator.

Lemma

Let $A \in B(L^2(\mathbb{R}^{2d}))$ and $z, z' \in \mathbb{R}^{2d}$.

1. If $T \in \mathcal{T}^p$ for $1 \leq p \leq \infty$, then $\|\alpha_z T\|_{\mathcal{T}^p} = \|T\|_{\mathcal{T}^p}$ and $\|\check{T}\|_{\mathcal{T}^p} = \|T\|_{\mathcal{T}^p}$.
2. $(\alpha_z A)^* = \alpha_z A^*$ and $\left(\check{A}\right)^* = (A^*)\check{}$.
3. $\pi(z)P = P\pi(-z)$, $\pi(\check{z}) = \pi(-z)$ and $(\alpha_z A)\check{} = \alpha_{-z}\check{A}$.

Vector-valued integration

We will need to integrate operator-valued functions $G : \mathbb{R}^{2d} \to B(L^2(\mathbb{R}^d))$ of the form $G(z) = g(z)F(z)$, where $g \in L^1(\mathbb{R}^{2d})$ and $F : \mathbb{R}^{2d} \to B(L^2(\mathbb{R}^d))$ is measurable, bounded and strongly continuous. The operator-valued integral $\iint_{\mathbb{R}^{2d}} g(z)F(z)\, dz \in B(L^2(\mathbb{R}^d))$ is defined in a weak and pointwise sense: for any $\psi \in L^2(\mathbb{R}^d)$ we define $\left(\iint_{\mathbb{R}^{2d}} g(z)F(z)\, dz\right)\psi$ by

$$\langle \left(\iint_{\mathbb{R}^{2d}} g(z)F(z)\, dz \right) \psi, \phi \rangle = \iint_{\mathbb{R}^{2d}} g(z)\langle F(z)\psi, \phi\rangle\, dz$$

for any $\phi \in L^2(\mathbb{R}^d)$. This defines an operator $\iint_{\mathbb{R}^{2d}} g(z)F(z)\, dz$, and we get the norm estimate $\| \iint_{\mathbb{R}^{2d}} g(z)F(z)\, dz\|_{B(L^2(\mathbb{R}^d))} \leq \|g\|_{L^1} \sup_{z\in\mathbb{R}^{2d}} \|F(z)\|_{B(L^2(\mathbb{R}^d))}$. For fixed $\psi \in L^2(\mathbb{R}^d)$ the $L^2(\mathbb{R}^d)$-valued function $z \mapsto g(z)F(z)\psi$ is even Bochner integrable.

Quantum variants of convolutions

- For $S, T \in \mathcal{T}^1$ we define the convolution of S and T

$$S * T(z) := \mathrm{tr}(S\alpha_z(\check{T})),$$

  where $\check{T} := PTP$ is the parity operator.
- For $f \in L^1(\mathbb{R}^{2d})$ and $S \in \mathcal{T}^1$ we define the convolution of f and S by

$$f * S := \iint_{\mathbb{R}^{2d}} f(y)\alpha_y(S)\, dy$$

Rank-one case

Given $\varphi_1, \varphi_2 \in L^2(\mathbb{R}^d)$ and let f be a function on $\mathbb{R}^{2d}$. Then

$$f * (\varphi_2 \otimes \varphi_1) = \iint_{\mathbb{R}^{2d}} f(z) V_{\varphi_1}(z) \pi(z) \varphi_2 \, dz.$$

is the a localization operator, denoted by $\mathcal{A}_f^{\varphi_1,\varphi_2}$.

Proof

Let $\psi \in L^2(\mathbb{R}^d)$ and set $S = \phi_2 \otimes \phi_1$.

$$\begin{aligned}(f * S)(\psi) &= \iint_{\mathbb{R}^{2d}} f(z)(\alpha_z S)(\psi) \, dz \\ &= \iint_{\mathbb{R}^{2d}} f(z) \langle \pi(z)^* \psi, \varphi_1 \rangle \pi(z) \varphi_2 \, dz \\ &= \iint_{\mathbb{R}^{2d}} f(z) V_{\varphi_1} \psi \, \pi(z) \varphi_2 \, dz = A_f^{\varphi_1,\varphi_2} \psi.\end{aligned}$$

Rank-one case

The Berezin transform of T with windows φ_1 and φ_2 is given by

$$\mathcal{B}^{\varphi_1,\varphi_2} T(z) = T * (\check{\varphi}_1 \otimes \check{\varphi}_2).$$

For $T = \psi_1 \otimes \psi_2$ we get that

$$(\psi_1 \otimes \psi_2) * (\check{\varphi}_1 \otimes \check{\varphi}_2) = V_{\varphi_1}\psi_1 \overline{V_{\varphi_2}\psi_2}.$$

We also note that the compact operators $K(L^2(\mathbb{R}^d))$ and the uniformly continuous functions vanishing at infinity, $C_0(\mathbb{R}^{2d})$, are corresponding under convolutions with trace class operators.

Lemma

Let $T \in \mathcal{T}^1$. If $f \in C_0(\mathbb{R}^{2d})$ and $S \in K(L^2(\mathbb{R}^d))$, then $f * T \in K(L^2(\mathbb{R}^d))$ and $S * T \in C_0(\mathbb{R}^{2d})$.

General Moyal idenity

Let $S, T \in \mathcal{T}^1$. The function $z \mapsto \mathrm{tr}(S\alpha_z T)$ for $z \in \mathbb{R}^{2d}$ is integrable and

$$\|\mathrm{tr}(S\alpha_z T)\|_{L^1} \leq \|S\|_{\mathcal{T}^1}\|T\|_{\mathcal{T}^1}.$$

Furthermore,

$$\iint_{\mathbb{R}^{2d}} \mathrm{tr}(S\alpha_z T)\, dz = \mathrm{tr}(S)\mathrm{tr}(T).$$

Remark

For rank-one operators $\psi_1 \otimes \psi_2$ and $\check{\varphi}_1 \otimes \check{\varphi}_2$ with ψ_i, φ_j in $L^2(\mathbb{R}^d)$ for $i, j \in \{1, 2\}$ the preceding equation gives the Moyal identity.

Proof

We start by showing the norm-inequality. First use the singular value decomposition of the operators S and T to write

$$S = \sum_{m\in\mathbb{N}} s_m \psi_m \otimes \phi_m \qquad T = \sum_{n\in\mathbb{N}} t_n \eta_n \otimes \xi_n,$$

where $\{s_m\}_{m\in\mathbb{N}}$ and $\{t_n\}_{n\in\mathbb{N}}$ are the singular values of S and T, respectively, and the sets $\{\psi_m\}_{m\in\mathbb{N}}, \{\phi_m\}_{m\in\mathbb{N}}, \{\eta_n\}_{n\in\mathbb{N}}$ and $\{\xi_n\}_{n\in\mathbb{N}}$ are orthonormal in $L^2(\mathbb{R}^d)$.

Then extend the set $\{\psi_m\}_{m\in\mathbb{N}}$ to an orthonormal basis $\{e_i\}_{i\in\mathbb{N}}$ of $L^2(\mathbb{R}^d)$. Using this basis to calculate the trace, we find that

$$\begin{aligned}
\operatorname{tr}(S\alpha_z T) &= \sum_{i\in\mathbb{N}} \langle S\pi(z)T\pi(z)^* e_i, e_i\rangle \\
&= \sum_{i,m,n\in\mathbb{N}} s_m t_n \langle \pi(z)^* e_i, \xi_n\rangle \langle \pi(z)\eta_n, \phi_m\rangle \langle \psi_m, e_i\rangle \\
&= \sum_{m,n\in\mathbb{N}} s_m t_n \langle \pi(z)^* \psi_m, \xi_n\rangle \langle \pi(z)\eta_n, \phi_m\rangle
\end{aligned}$$

Proof-ctd.

By Moyal's identity, $V_{\xi_n}\psi_m, V_{\eta_n}\phi_m \in L^2(\mathbb{R}^{2d})$, and so $V_{\xi_n}\psi_m \overline{V_{\eta_n}\phi_m} \in L^1(\mathbb{R}^{2d})$ by Hölder's inequality. The following computation shows that the series above converges absolutely in $L^1(\mathbb{R}^d)$ with the desired norm estimates.

$$
\begin{aligned}
\| \sum_{m,n\in\mathbb{N}} s_m t_n V_{\xi_n}\psi_m \overline{V_{\eta_n}\phi_m} \|_{L^1} &\leq \sum_{m,n\in\mathbb{N}} s_m t_n \| V_{\xi_n}\psi_m \overline{V_{\eta_n}\phi_m} \|_{L^1} \\
&\leq \sum_{m,n\in\mathbb{N}} s_m t_n \| V_{\xi_n}\psi_m \|_{L^2} \| V_{\eta_n}\phi_m \|_{L^2} \\
&= \sum_{m,n\in\mathbb{N}} s_m t_n \|\xi_n\|_{L^2} \|\psi_m\|_{L^2} \|\eta_n\|_{L^2} \|\phi_m\|_{L^2} \\
&= \sum_{m,n\in\mathbb{N}} s_m t_n = \|S\|_{\mathcal{T}^1} \|T\|_{\mathcal{T}^1}.
\end{aligned}
$$

Proof-ctd.

The equality $\iint_{\mathbb{R}^{2d}} \mathrm{tr}(S\alpha_z T)\,dz = \mathrm{tr}(S)\mathrm{tr}(T)$ now follows easily from Moyal's identity.

$$\begin{aligned}\iint_{\mathbb{R}^{2d}} \mathrm{tr}(S\alpha_z T)\,dz &= \iint_{\mathbb{R}^{2d}} \sum_{m,n\in\mathbb{N}} s_m t_n V_{\xi_n}\psi_m \overline{V_{\eta_n}\phi_m}\,dz \\ &= \sum_{m,n\in\mathbb{N}} s_m t_n \iint_{\mathbb{R}^{2d}} V_{\xi_n}\psi_m \overline{V_{\eta_n}\phi_m}\,dz \\ &= \sum_{m,n\in\mathbb{N}} s_m t_n \langle \psi_m, \phi_m\rangle\langle \eta_n, \xi_n\rangle \\ &= \mathrm{tr}(S)\mathrm{tr}(T),\end{aligned}$$

where the last equality follows from an easy calculation of $\mathrm{tr}(S)$ and $\mathrm{tr}(T)$.

Theorem

The convolution operations are commutative and associative.

proof

Commutativity: Let $S, T \in \mathcal{T}^1$. We find that

$$\begin{aligned} S * T(z) &= \mathrm{tr}(S\alpha_z \check{T}) \\ &= \mathrm{tr}(S\pi(z)PTP\pi(z)^*) \\ &= \mathrm{tr}(T(\alpha_{-z}S)\check{}\,) \\ &= \mathrm{tr}(T\alpha_z\check{S}) = T * S(z) \end{aligned}$$

We have made extensive use of the property $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.

Associativity–Proof

The most interesting case is the convolution of three operators. We will need the general Moyal identity in addition to some more technical calculations. Let $T_1, T_2, T_3 \in \mathcal{T}^1$.
To show that $T_1 * (T_2 * T_3) = (T_1 * T_2) * T_3$ it will be helpful to assume an arbitrary operator $T_0 \in \mathcal{T}^1$. If we can show that the dual space actions $\langle T_1 * (T_2 * T_3), T_0 \rangle = \langle (T_1 * T_2) * T_3, T_0 \rangle$ for any T_0, we will have shown that the two expressions define the same element in the dual space $B(L^2(\mathbb{R}^d))$, and therefore the same operator. It will suffice to show that

$$\mathrm{tr}\left[T_0(T_1 * (T_2 * T_3))\right] = \mathrm{tr}\left[T_0((T_1 * T_2) * T_3)\right].$$

Associativity–Proof

Writing out the left side of the equation, we find that

$$\begin{aligned}
\operatorname{tr}\left[T_0(T_1 * (T_2 * T_3))\right] &= \operatorname{tr}\left[T_0 \iint_{\mathbb{R}^{2d}} \operatorname{tr}(T_2 \alpha_x \check{T}_3)\alpha_x T_1 \, dx\right] \\
&= \iint_{\mathbb{R}^{2d}} \operatorname{tr}\left[T_2 \alpha_x \check{T}_3\right] \operatorname{tr}\left[(\alpha_x T_1) T_0\right] \, dx \\
&= \iint_{\mathbb{R}^{2d}} \iint_{\mathbb{R}^{2d}} \operatorname{tr}\left[(\alpha_x T_1) T_0 \alpha_y (T_3 \alpha_x \check{T}_2)\right] \, dy \, dx.
\end{aligned}$$

The last equality uses the general Moyal identity to introduce the second integral, and also exploits the commutativity of convolutions to switch the order of T_2 and T_3. It is a simple exercise to check that $\alpha_y(AB) = (\alpha_y A)(\alpha_y B)$ for operators A and B, hence $\alpha_y(T_3 \alpha_x \check{T}_2) = (\alpha_y T_3)(\alpha_x \alpha_y \check{T}_2)$.

Associativity–Proof

Using this in our calculation we get that

$$\begin{aligned}
&\iint_{\mathbb{R}^{2d}} \iint_{\mathbb{R}^{2d}} \operatorname{tr}\left[(\alpha_x T_1) T_0 \alpha_y (T_3 \alpha_x \check{T}_2)\right] \; dy \, dx \\
&= \iint_{\mathbb{R}^{2d}} \iint_{\mathbb{R}^{2d}} \operatorname{tr}\left[(\alpha_x T_1) T_0 (\alpha_y T_3)(\alpha_x \alpha_y \check{T}_2))\right] \; dy \, dx \\
&= \iint_{\mathbb{R}^{2d}} \iint_{\mathbb{R}^{2d}} \operatorname{tr}\left[(T_0 \alpha_y T_3)(\alpha_x \alpha_y \check{T}_2)(\alpha_x T_1)\right] \; dy \, dx.
\end{aligned}$$

As above, $(\alpha_x \alpha_y \check{T}_2)(\alpha_x T_1) = \alpha_x((\alpha_y \check{T}_2) T_1)$. We may use Fubini's theorem to change the order of integration, and then invoke the general Moyal idenity again to reduce the expression to a form that we recognize as the desired equality.

Associativity–Proof

$$\iint_{\mathbb{R}^{2d}} \iint_{\mathbb{R}^{2d}} \operatorname{tr}\left[(T_0 \alpha_y T_3)\alpha_x((\alpha_y \check{T}_2)T_1)\right] \, dx \, dy$$

$$= \iint_{\mathbb{R}^{2d}} \operatorname{tr}\left[T_0 \alpha_y T_3\right] \operatorname{tr}\left[(\alpha_y \check{T}_2)T_1\right] \, dy$$

$$= \operatorname{tr}\left[T_0((T_1 * T_2) * T_3)\right].$$

Using duality we can extend the domains of the convolutions introduced above, by allowing one factor to belong to the dual space. For instance, if $h \in L^\infty(\mathbb{R}^{2d})$ and $S \in \mathcal{T}^1$, we define $h * S \in B(L^2(\mathbb{R}^d))$ by $\langle h * S, T \rangle = \langle h, T * \check{S}^* \rangle$ for every $T \in \mathcal{T}^1$. A standard interpolation argument then gives the following result, since $(L^1(\mathbb{R}^d), L^\infty(\mathbb{R}^d))_\theta = L^p$ and $(\mathcal{T}^1, \mathcal{T}^\infty)_\theta = \mathcal{T}^p$ with $\frac{1}{p} = 1 - \theta$.

Proposition

Let $1 \leq p, q, r \leq \infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$. If $f \in L^p(\mathbb{R}^{2d}), g \in L^q(\mathbb{R}^{2d}), S \in \mathcal{T}^p$ and $T \in \mathcal{T}^q$, then the following convolutions may be defined and satisfy the norm estimates

$$\|f * T\|_{\mathcal{T}^r} \leq \|f\|_{L^p} \|T\|_{\mathcal{T}^q},$$
$$\|g * S\|_{\mathcal{T}^r} \leq \|g\|_{L^q} \|S\|_{\mathcal{T}^p},$$
$$\|S * T\|_{L^r} \leq \|S\|_{\mathcal{T}^p} \|T\|_{\mathcal{T}^q}.$$

Here is a more abstract aspect of the convolution of a function with an operator.

Theorem

$\mathcal{T}^p$ is a Banach module over $L^1(\mathbb{R}^{2d})$ for the module action given by $(f, S) \mapsto f \star S$ for $1 \leq p \leq \infty$ since $\|f * T\|_{\mathcal{T}^p} \leq \|f\|_{L^1} \|T\|_{\mathcal{T}^p}$.

Cohen-Hewitt factorization yields

Factorization

For $1 \leq p < \infty$, every element of $\mathcal{T}^p$ can be written as $f * S$ for $f \in L^1(\mathbb{R}^{2d})$, $S \in \mathcal{T}^p$.

We will now introduce an analogue of the Fourier transform for a trace class operator S.

Fourier-Wigner transform

The **Fourier- Wigner transform** $\mathcal{F}_W S$ of S is the function given by

$$\mathcal{F}_W S(z) = e^{-\pi i x \cdot \omega} \operatorname{tr}(\pi(-z)S)$$

for $z \in \mathbb{R}^{2d}$.

Example

Let $S = \varphi_2 \otimes \varphi_1$ with $\varphi_1, \varphi_2 \in L^2(\mathbb{R}^d)$. The Fourier-Wigner transform of S is given by

$$\mathcal{F}_W(\varphi_2 \otimes \varphi_1)(z) = A(\varphi_2, \varphi_1)(z),$$

where $A(\varphi_2, \varphi_1)(z)$ is the cross-ambiguity function.

Gaussian

Consider the Gaussian $\varphi(t) = 2^{d/4} e^{-\pi t \cdot t}$ for $t \in \mathbb{R}^d$ and the operator $S = \varphi \otimes \varphi$. We know that $\mathcal{F}_W S = e^{\pi i x \cdot \omega} V_\varphi \varphi(z)$, and then find that $\mathcal{F}_W(\varphi \otimes \varphi)(z) = e^{2\pi i x \cdot \omega} e^{-\frac{1}{2}\pi z \cdot z}$.

Another way of associating an operator to a function is to define the operator as a superposition of time-frequency shifts using the theory of vector-valued integration. The *integrated Schrödinger representation* is the map $\rho : L^1(\mathbb{R}^{2d}) \to B(L^2(\mathbb{R}^d))$ given by

$$\rho(f) = \iint_{\mathbb{R}^{2d}} f(z) e^{-\pi i x \cdot \omega} \pi(z)\, dz,$$

where the integral is defined in the weak and pointwise sense. We say that f is the *twisted Weyl symbol* of $\rho(f)$.
We will use the important product formula $\rho(f)\rho(g) = \rho(f \natural g)$, where the product $\natural$ is the *twisted convolution*, defined by

$$f \natural g(z) = \iint_{\mathbb{R}^{2d}} f(z - z') g(z') e^{\pi i \sigma(z, z')}\, dz'$$

for $f, g \in L^1(\mathbb{R}^{2d})$

Fact

ρ may be extended to a unitary operator from $L^2(\mathbb{R}^{2d})$ to $\mathcal{T}^2$, and that the twisted convolution $f \natural g$ may be defined for $f, g \in L^2(\mathbb{R}^{2d})$ with norm estimate $\|f \natural g\|_{L^2} \leq \|f\|_{L^2} \|g\|_{L^2}$.

Proposition

The Fourier-Wigner transform extends to a unitary operator $\mathcal{F}_W : \mathcal{T}^2 \to L^2(\mathbb{R}^{2d})$. This extension is the inverse operator of the integrated Schrödinger representation ρ, and $\mathcal{F}_W(ST) = \mathcal{F}_W(S) \natural \mathcal{F}_W(T)$ for $S, T \in \mathcal{T}^2$.

The Fourier-Wigner transform shares several properties with the Fourier transform of functions.

Riemann-Lebesgue lemma

If $S \in \mathcal{T}^1$, the Fourier-Wigner transform $\mathcal{F}_W(S)$ is continuous and vanishes at infinity, i.e. $\lim_{|z|\to\infty} |\mathcal{F}_W(z)| = 0$.

Mapping properties

Let $f \in L^1(\mathbb{R}^{2d})$ and $S, T \in \mathcal{T}^1$.

1. $\mathcal{F}_\sigma(S * T) = \mathcal{F}_W(S)\mathcal{F}_W(T)$.
2. $\mathcal{F}_W(f * S) = \mathcal{F}_\sigma(f)\mathcal{F}_W(S)$.

Proof of item 1

By definition,
$\mathcal{F}_\sigma(S * T)(z) = \iint_{\mathbb{R}^{2d}} \operatorname{tr}\left[S\pi(z')\check{T}\pi(z')^*\right] e^{-2\pi i\sigma(z,z')}\, dz'$. Since $e^{-2\pi i\sigma(z,z')}\pi(z') = \alpha_{-z}\pi(z')$, the integrand may be written in a way that will allow us to use the general Moyal identity:

$$\begin{aligned} \operatorname{tr}\left[S\pi(z')\check{T}\pi(z)^*\right] e^{-2\pi i\sigma(z,z')} &= \operatorname{tr}\left[Se^{-2\pi i\sigma(z,z')}\pi(z')\check{T}\pi(z')^*\right] \\ &= \operatorname{tr}\left[S\pi(-z)\pi(z')\pi(-z)^*\check{T}\pi(z')^*\right]. \end{aligned}$$

The general Moyal identity and
$\operatorname{tr}(\pi(z)\check{T}) = \operatorname{tr}(\pi(z)PTP) = \operatorname{tr}(P\pi(z)PT) = \operatorname{tr}(\pi(-z)T)$ gives

$$\begin{aligned} \mathcal{F}_\sigma(S * T)(z) &= \iint_{\mathbb{R}^{2d}} \operatorname{tr}\left[S\pi(-z)\alpha_{z'}(\pi(-z)^*\check{T})\right]\, dz' \\ &= \operatorname{tr}(S\pi(-z))\operatorname{tr}(\pi(-z)^*\check{T}) \\ &= \operatorname{tr}(S\pi(-z))\operatorname{tr}(e^{-2\pi i x\cdot\omega}\pi(z)\check{T}) \\ &= \operatorname{tr}(e^{-\pi i x\cdot\omega}S\pi(-z))\operatorname{tr}(e^{-\pi i x\cdot\omega}\pi(-z)T) \\ &= \mathcal{F}_W(S)(z)\mathcal{F}_W(T)(z). \end{aligned}$$

Proof of item 2

By taking the trace inside the integral:

$$\begin{aligned}\mathcal{F}_W(f * S)(z) &= e^{-\pi i x\cdot\omega}\mathrm{tr}\left(\pi(-z)\iint_{\mathbb{R}^{2d}} f(z')\pi(z')S\pi(z')^*\,dz'\right)\\ &= e^{-\pi i x\cdot\omega}\iint_{\mathbb{R}^{2d}} f(z')\mathrm{tr}\left[\pi(-z)\pi(z')S\pi(z')^*\right]\,dz'.\end{aligned}$$

A simple manipulation of the integrand yields that $\mathrm{tr}\left[\pi(-z)\pi(z')S\pi(z')^*\right] = e^{-2\pi i\sigma(z,z')}\mathrm{tr}(\pi(-z)S)$. Inserting this expression into our calculation concludes the proof, since

$$\begin{aligned}\mathcal{F}_W(f * S)(z) &= e^{-\pi i x\cdot\omega}\iint_{\mathbb{R}^{2d}} f(z')e^{2\pi i\sigma(z,z')}\mathrm{tr}(\pi(-z)S)\,dz'\\ &= e^{-\pi i x\cdot\omega}\mathrm{tr}(\pi(-z)S)\iint_{\mathbb{R}^{2d}} f(z')e^{-2\pi i\sigma(z,z')}\,dz'\\ &= \mathcal{F}_\sigma(f)\mathcal{F}_W(S).\end{aligned}$$

Hausdorff-Young inequality

Let $1 \leq p \leq 2$ and let q be the conjugate exponent determined by $\frac{1}{p} + \frac{1}{q} = 1$. If $S \in \mathcal{T}^p$, then $\mathcal{F}_W(S) \in L^q(\mathbb{R}^{2d})$ with norm estimate

$$\|\mathcal{F}_W(S)\|_{L^q} \leq \|S\|_{\mathcal{T}^p}.$$

Lieb's inequality

If we pick $S = \psi \otimes \phi$ for $\psi, \phi \in L^2(\mathbb{R}^d)$ in the Hausdorff-Young inequality, we obtain for $2 \leq q < \infty$ that

$$\iint_{\mathbb{R}^{2d}} |V_\phi \psi(z)|^q \, dz \leq \|\psi\|_{L^2}^q \|\phi\|_{L^2}^q.$$

This is Lieb's inequality, except for the constant $\left(\frac{2}{q}\right)^d$ that makes Lieb's inequality sharp. Hence we can consider Lieb's uncertainty principle to be a sharp version of the Hausdorff-Young inequality for rank-one operators.

As a corollary, we note an extension of Lieb's uncertainty principle to trace class operators.

Hausdorff-Young–sharp

Let $2 \leq q < \infty$. If $S \in \mathcal{T}^1$, then

$$\|\mathcal{F}_W(S)\|_{L^q} \leq \left(\frac{2}{q}\right)^{d/q} \|S\|_{\mathcal{T}^1}.$$

Observation

Lieb's inequality expresses localization results of a function ϕ with respect to a window function ψ in terms of summability conditions on the the STFT, a time-frequency representation.

Modulation spaces

The modulation spaces are a class of spaces of functions and distributions introduced by Feichtinger in a series of papers starting with the introduction of the so-called Feichtinger algebra in , and they have since been recognized as a suitable setting for time-frequency analysis.

STFT-extenstion

The STFT can be extended to other spaces by interpreting the bracket $\langle \cdot, \cdot \rangle$ as a duality bracket. This allows us to consider $V_\phi \psi$ for $\phi \in \mathcal{S}(\mathbb{R}^d)$ and $\psi \in \mathcal{S}'(\mathbb{R}^d)$.

Modulation spaces - definition

Fix a window $\phi \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$. For $1 \leq p, q \leq \infty$, the **modulation space** $M_m^{p,q}(\mathbb{R}^d)$ is the set of tempered distributions ψ such that

$$\|\psi\|_{M_m^{p,q}} = \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |V_\phi \psi(x, \omega)|^p m(x, \omega) \, dx \right)^{q/p} d\omega \right)^{1/q} < \infty,$$

where m is positive function (weight) on $\mathbb{R}^{2d}$.
In the special cases where p or q is ∞, the integral is replaced by an essential supremum. When $p = q$, we will denote the space $M_m^{p,p}(\mathbb{R}^d)$ by $M_m^p(\mathbb{R}^d)$.

Remark

The modulation spaces are Banach spaces with the norms $\|\psi\|_{M_m^{p,q}}$, and using a different window $\phi \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$ in the definition yields the same spaces with equivalent norms.

Modulation spaces - examples

- $p = q = 1$ and $m = 1$ is *Feichtinger's algebra* $M^1(\mathbb{R}^d)$ also denoted by $S_0(\mathbb{R}^d)$, a useful space of test functions.
- $p = q = 2$ and $m(x, \omega) = (1 + |\omega|^2)^{s/2}$ gives the *Sobolev spaces* $H^2_s(\mathbb{R}^d)$.
- $p = q = 2$ and $m(x, \omega) = (1 + |x|^2)^{s/2}$ gives the *Bessel potential spaces* $W^2_s(\mathbb{R}^d)$.
- $p = q = 2$ and $m(x, \omega) = (1 + |x|^2 + |\omega|^2)^{s/2}$ gives the *Shubin class* $Q^2_s(\mathbb{R}^d)$.
- $p = \infty$ and $q = 1$ and $m(x, \omega) = (1 + |x|^2 + |\omega|^2)^{s/2}$ is often called *Sjöstrand's class*.
- $p = q = \infty$ and $m = 1$ is the dual of *Feichtinger's algebra* often denoted by $S_0'(\mathbb{R}^d)$, a useful space of distributions.

Modulation spaces have turned out to be of good use in the theory of pseudodifferential operators, Schrödinger equations, Fourier integral operators, Fourier multipliers, etc. .

Cross-Wigner distribution

The **cross-Wigner distribution**, $W(\psi, \phi)$, of two functions ψ and ϕ on $\mathbb{R}^d$ is by definition given by

$$W(\psi, \phi)(x, \omega) = \int_{\mathbb{R}^d} \psi\left(x + \frac{t}{2}\right) \overline{\phi\left(x - \frac{t}{2}\right)} e^{-2\pi i \omega \cdot t}\, dt.$$

This expression is similar to the definition of the cross-ambiguity function, and in fact $W(\psi, \phi) = \mathcal{F}_\sigma A(\psi, \phi)$.

Weyl calculus

Our main motivation for studying the cross-Wigner distribution is its connection with the **Weyl calculus.** For σ in the space of tempered distributions $\mathcal{S}'(\mathbb{R}^{2d})$ and $\psi, \phi \in \mathcal{S}(\mathbb{R}^d)$, we define the *Weyl transform* L_σ of σ to be the operator given by

$$\langle L_\sigma \psi, \phi \rangle = \langle \sigma, W(\phi, \psi) \rangle.$$

σ is called the *Weyl symbol* of the operator L_σ.

Recall the **integrated Schrödinger representation**, which is the map $\rho : L^1(\mathbb{R}^{2d}) \to B(L^2(\mathbb{R}^d))$ given by

$$\rho(f) = \iint_{\mathbb{R}^{2d}} f(z) e^{-\pi i x \cdot \omega} \pi(z)\, dz,$$

Observation

The relationship between the Weyl calculus and the integrated Schrödinger representation is given by $L_f = \rho(\mathcal{F}_\sigma f)$ for a symbol f.

One may assign to a function k on $\mathbb{R}^{2d}$ a so-called **integral operator** A_k on $L^2(\mathbb{R}^d)$ by

$$A_k \psi(s) = \int_{\mathbb{R}^d} k(s,t) \psi(t)\, dy \qquad (3)$$

for $\psi \in L^2(\mathbb{R}^d)$. k is called the **kernel** of A_k.

Useful class of A_k's

We will let $\mathcal{M}$ denote the set of integral operators A_k with kernel k in $M^1(\mathbb{R}^{2d})$.

Lemma

$\mathcal{M}$ is also the set of operators with Weyl symbol or twisted Weyl symbol in $M^1(\mathbb{R}^{2d})$

Concrete realization $\mathcal{M}$

Let $k \in M^1(\mathbb{R}^{2d})$ and let A_k be the integral operator with kernel k. Let $\{w_n\}_{n\in\mathbb{N}}$ be a Wilson basis for $L^2(\mathbb{R}^d)$, and denote by W_{mn} the corresponding Wilson basis for $L^2(\mathbb{R}^{2d})$ given by
$W_{mn}(x,y) = w_m(x)\overline{w_n(y)}$.
Then $A_k \in \mathcal{T}^1$ with $\|A_k\|_{\mathcal{T}^1} \leq C\|k\|_{M^1}$ for some constant C, and $A_k = \sum_{m,n\in\mathbb{N}} \langle k, W_{mn}\rangle w_m \otimes w_n$ where the sum converges in the $\mathcal{T}^1$-norm

Lemma

Let $f \in L^1(\mathbb{R}^{2d})$, and let L_f be the Weyl transform of f.

- $\alpha_z(L_f) = L_{T_z f}$ for $z \in \mathbb{R}^{2d}$.
- $\check{L_f} = L_{\check{f}}$.
- $L_f^* = L_{f^*}$.

In particular, if $S \in \mathcal{M}$, then $\alpha_z(S), \check{S}, S^* \in \mathcal{M}$.

Remark

Hence, applying α_z to a pseudodifferential operator amounts to a translation of its symbol.

Lemma

Let $f \in L^1(\mathbb{R}^{2d})$ and $S \in \mathcal{T}^1$. The twisted Weyl symbol of $f * S$ is the function $\mathcal{F}_\sigma(f)\mathcal{F}_W(S)$. In particular, if $\varphi_1, \varphi_2 \in L^2(\mathbb{R}^d)$, then the twisted Weyl symbol of the localization operator $\mathcal{A}_f^{\varphi_1,\varphi_2}$ is the function $\mathcal{F}_\sigma(f) \cdot A(\varphi_2, \varphi_1)$.