On geometrical structures, associated with linear differential operators

Valentin Lychagin

The Arctic University of Norway, Tromso & Institute of Control Science, Russian Academy of Science, Moscow

July 01, 2019

The second order operators

$$A = a_{11}\frac{\partial^2}{\partial x^2} + 2a_{12}\frac{\partial^2}{\partial x \partial y} + a_{22}\frac{\partial^2}{\partial y^2} + a_1\frac{\partial}{\partial x} + a_2\frac{\partial}{\partial y} + a_0,$$

a_{ij}, a_i are smooth functions in x and y.

Observation

- $X, Y-$ vector fields $\Longrightarrow X \circ Y \in \mathbf{Diff}_2$ 2nd order operator$\Longrightarrow X \circ Y \bmod \mathbf{Diff}_1$ behaves like symmteric tensor.
- $X \circ Y \bmod \mathbf{Diff}_1 = X \cdot Y \in \mathbf{S}^2 T$
- $$\sigma_A = A \bmod \mathbf{Diff}_1 \in \mathbf{S}^2 T$$
- $$\sigma_A = a_{11}\partial_x \cdot \partial_x + 2a_{12}\partial_x \cdot \partial_y + a_{22}\partial_y \cdot \partial_y \in \mathbf{S}^2 T.$$

Hamiltonian

- Symbol $\sigma_A -$ quadtric on $T^* -$

$$H_A = a_{11} p_x^2 + 2a_{12} p_x p_y + a_{22} p_y^2$$

  in the canonical coordinates (x, y, p_x, p_y) .

- Let $a = \det \|a_{ij}\| \neq 0 \Longrightarrow \|g_{ij}\| = \|a_{ij}\|^{-1} \Longrightarrow$

$$g_A = g_{11} dx^2 + 2g_{12} dxdy + g_{22} dy^2$$

  -quadric on $T \Longrightarrow$metric, associated with the operator.

Universal Decomposition

- Δ_A- the Beltrami-Laplace operator, associated with metric $g_A \Longrightarrow A - \Delta_A \in \mathbf{Diff}_1$
- Decomposition
$$A = \Delta_A + v_A + s_A,$$
where v_A-vector field, $s_A = A(1) -$ function.
- Coordinates
$$\begin{aligned} \Delta_A &= \sum_{i,j} \left(\partial_i a_{ij} \partial_j - \frac{1}{2} a_{ij} \partial_j \left(\ln \left(|a| \right) \right) \partial_j \right), \\ v_A &= \sum_i \left(a_i - \sum_j \left(\partial_j \left(a_{ij} \right) - \frac{1}{2} a_{ij} \partial_j \left(\ln \left(|a| \right) \right) \right) \right) \partial_i, \\ s_A &= a_0. \end{aligned}$$

Naturality

- $\phi-$ diffeomorphism and $\phi_*(f) = f \circ \phi^{-1}$, $\phi_*(A) = \phi_* \circ A \circ \phi_*^{-1}, \ldots$.
- Naturality:

$$\left\Vert \begin{array}{c} A \\ \sigma_A \left(\text{or } g_A\right) \\ \Delta_A \\ v_A \\ s_A \end{array} \right\Vert \xrightarrow{\phi_*} \left\Vert \begin{array}{c} \phi_*(A) \\ \phi_*(\sigma_A) = \sigma_{\phi_*(A)} \\ \phi_*(\Delta_A) = \Delta_{\phi_*(A)} \\ \phi_*(v_A) = v_{\phi_*(A)} \\ \phi_*(s_A) = s_{\phi_*(A)} \end{array} \right\Vert$$

- Scalar invariants of operators:

$$s_A, \ k_A = \text{curvature of } g_A, \ g_A(v_A, v_A), \ v_A(s_A), \ldots.$$

Classification, natural classes

- Constant coefficients $\Longleftrightarrow$

$$k_A = 0, ds_A = 0, d_\nabla (v_A) = 0,$$

where d_∇ is the covariant differential wrt Levi-Civita connection, associated with g_A.

- Vector field v_A is an *isometry* of g_A, $L_{v_A}(g_A) = 0$, and $k_A = 0$, then (for elliptic case)

$$A = \partial_x^2 + \partial_y^2 + (\alpha_1 - \gamma_1 y)\,\partial_x + (\alpha_2 + \gamma_1 x)\,\partial_y + a_0.$$

- Vector field v_A is an *similarity* of $g_A, L_{v_A}(g_A) = \lambda g_A$, and $k_A = 0$, then (for elliptic case)

$$A = \partial_x^2 + \partial_y^2 + (\alpha_1 - \gamma_1 y + \gamma_2 x)\,\partial_x + (\alpha_2 + \gamma_1 x + \gamma_2 y)\,\partial_y + a_0.$$

Here $\lambda, \alpha_i, \gamma_i$ are constants.

- etc,....

Classification, models

- *Natural coordinates*: two invariants J_1, J_2, such that functions $X = J_1(A)$ and $Y = J_2(A)$ are independent (in a domain).
- *Model, associated with invariants* J_1, J_2, =operator A in coordinates X, Y

$$A = A_{11}\frac{\partial^2}{\partial X^2} + 2A_{12}\frac{\partial^2}{\partial X \partial Y} + A_{22}\frac{\partial^2}{\partial Y^2} + A_1\frac{\partial}{\partial X} + A_2\frac{\partial}{\partial Y} + A_0$$

- Two operators are equivalent wrt diffeomorphism group if and only if their models coincide.

The third order differential operators

$$
\begin{aligned}
A \;=\;& a_{111}\frac{\partial^3}{\partial x^3} + 3a_{112}\frac{\partial^3}{\partial x^2 \partial y} + 3a_{122}\frac{\partial^3}{\partial x \partial y^2} + a_{222}\frac{\partial^3}{\partial y^3} + \\
& a_{11}\frac{\partial^2}{\partial x^2} + 2a_{12}\frac{\partial^2}{\partial x \partial y} + a_{22}\frac{\partial^2}{\partial y^2} + a_1\frac{\partial}{\partial x} + a_2\frac{\partial}{\partial y} + a_0
\end{aligned}
$$

- Symbol

$$\sigma_A = A \bmod \mathbf{Diff}_2,$$

$$\sigma_A = a_{111}\partial_x^3 + 3a_{112}\partial_x^2 \cdot \partial_y + 3a_{122}\partial_x \cdot \partial_y^2 + a_{222}\partial_y^3 \in \mathbf{S}^3 T,$$

or

$$H = a_{111}\; p_x^3 + 3a_{112}\; p_x^2 p_y + 3a_{122}\; p_x p_y^2 + a_{222}\; p_y^3.$$

Types of operators

- **Regularity**: equation $\sigma_A = 0$ has distinct roots (distinct characteristics), $D_A \neq 0$, where D_A is the discriminant:

$$\begin{aligned} D_A &= a_{111}a_{112}a_{122}a_{222} - 4\left(a_{111}a_{122}^3 + a_{222}a_{112}^3\right) \\ &\quad +3a_{112}^2a_{122}^2 - a_{111}^2a_{222}^2 \end{aligned}$$

- **Hyperbolic** case= three distinct real roots, $D_A > 0, \implies \sigma_A = X_1 \cdot X_2 \cdot X_3$, where $X_i \neq 0$ are pairwise linear independent vector fields.
- Hyperbolic case: there are local coordinates (x, y) , such that

$$\sigma_A = \partial_x \cdot \partial_y \cdot (a\partial_x + b\partial_y)\,,$$

  where $ab \neq 0$.
- **Ultrahyperbolic** case= one real root and two complex roots, $D_A < 0, \implies \sigma_A = X_1 \cdot \left(X_2^2 + X_3^2\right)$.
- Ultrahyperbolic case:there are local coordinates (x, y) , such that

$$\sigma_A = (a\partial_x + b\partial_y) \cdot \left(\partial_x^2 + \partial_y^2\right),$$

  where $a^2 + b^2 \neq 0$

Wagner connection

- For any regular operator A there is and unique affine connection ∇^A on the plane, such that the parallel transports preserves the symbol σ_A, (Wagner, 1938).
- Properties of the Wagner connection:

1. Naturality:
$$\nabla^{\phi_*(A)} = \phi_* \left(\nabla^A \right),$$
for any differeomorphism ϕ.
2. The curvature of the Wagner connection is zero.

Coordinates

- Christoffel symbols (hyperbolic case):

$$\Gamma^1_{11} = \frac{1}{3}\left(\ln\frac{b}{a^2}\right)_x, \Gamma^2_{22} = \frac{1}{3}\left(\ln\frac{a}{b^2}\right)_y,$$

$$\Gamma^1_{12} = \frac{1}{3}\left(\ln\frac{b}{a^2}\right)_y, \Gamma^2_{21} = \frac{1}{3}\left(\ln\frac{a}{b^2}\right)_x.$$

- Christoffel symbols (ultrahyperbolic case):

$$\Gamma^1_{11} = \Gamma^2_{21} = -\frac{1}{6}\left[\ln\left(a^2+b^2\right)\right]_x, \Gamma^1_{12} = \Gamma^2_{22} = -\frac{1}{6}\left[\ln\left(a^2+b^2\right)\right]_y,$$

$$\Gamma^1_{21} = -\Gamma^2_{11} = \frac{a_x b - a b_x}{a^2+b^2}, \Gamma^1_{22} = -\Gamma^2_{12} = \frac{a b_y - a_y b}{a^2+b^2}.$$

Group-type symbols

- The bracket
$$X, Y \longmapsto T^A(X, Y),$$
where T^A is the torsion tensor of the Wagner connection, defines a Lie algebra structure on vector space $\mathfrak{g}_A$ of ∇^A-constant vector fields, if T^A is ∇^A-constant itself.
- Symbol σ_A is locally equivalent to the symbol with constant coefficients if and only if the Lie algebra $\mathfrak{g}_A$ is commutative, or $T^A = 0$.
- Symbol σ_A is locally equivalent to the symbol
$$\begin{aligned}\sigma &= c_1 \exp(3y)\, p_x^3 + 3c_2 \exp(2y)\, p_x^2 p_y \\ &\quad + 3c_3 \exp(y)\, p_x p_y^2 + c_4 p_y^3,\end{aligned}$$
where $c_i \in \mathbb{R}$, if and only if the Lie algebra $\mathfrak{g}_A$ is solvable, i.e.
$$T^A \neq 0, \text{ but } d_{\nabla^A}\left(T^A\right) = 0.$$

Quantizations

- Quantization= morphism from symbols to operators: $Q = Q_\nabla : \sigma \to \widehat{\sigma}$.
- Higher order differentials:
$$d_\nabla^k : C^\infty(M) \to S^k(T^*)(M),$$
where ∇ is an affine connection on M :
$$d_\nabla^1(f) = df, \;\; d_\nabla^2(f) = \mathrm{Sym}(d_\nabla(df)), \; d_\nabla^3(f) = \mathrm{Sym}\left(d_\nabla\left(d_\nabla^2 f\right)\right).$$
- $$Q(f) = f, \; Q(X) = X,$$
if $\sigma = X$ is a vector field.
- $$Q(\sigma_2)(f) = \frac{1}{2}\left\langle \sigma_2, d_\nabla^2(f)\right\rangle,$$
if $\sigma_2 \in S^2 T$.
- $$Q(\sigma_3)(f) = \frac{1}{6}\left\langle \sigma_3, d_\nabla^3(f)\right\rangle,$$

Universal decomposition of 3rd order operators

- $$\begin{aligned} A &\Longrightarrow \sigma_3(A) = \mathrm{smbl}(A) \in S^3T \Longrightarrow \widehat{\sigma_3} \in \mathbf{Diff}_3 \Longrightarrow \\ A - \widehat{\sigma_3} &\in \mathbf{Diff}_2 \Longrightarrow \sigma_2 = \mathrm{smbl}(A - \widehat{\sigma_3}) \Longrightarrow \widehat{\sigma_2} \in \mathbf{Diff}_2 \Longrightarrow \\ A - \widehat{\sigma_3} - \widehat{\sigma_2} &\in \mathbf{Diff}_1 \Longrightarrow \\ A &= Q(\sigma_3 + \sigma_2 + \sigma_1 + \sigma_0), \end{aligned}$$

  where $\sigma_3 \in S^3T, \sigma_2 \in S^2T, \sigma_1 \in T$, σ_0-smooth function.
- If $Q = Q_{\nabla^A}$ is the quantization, associated with the Wagner connection, then decomposition

  $$A = Q_{\nabla^A}(\sigma_3(A) + \sigma_2(A) + \sigma_1(A) + \sigma_0(A))$$

  is the invariant wrt the diffeomorphism group.

Wagner quantization

- Hyperbolic case:

$$\sigma_h = 3\exp(a+b)\left(\exp(b)\partial_x + \exp\left(a\right)\partial_y\right)\cdot\partial_x\cdot\partial_y .$$

- Christoffel coefficients:

$$\Gamma^1_{11} = -b_x, \Gamma^1_{12} = -b_y, \Gamma^2_{21} = -a_x, \Gamma^2_{22} = -a_y,$$

Wagner quantization

- Quantization, $k = 3$.

$$\begin{aligned}\widehat{\sigma_h} &= 3\exp(a+b)\left(\exp(b)\partial_x + \exp(a)\partial_y\right)\partial_x\partial_y + \\ &\quad 3\exp(a+2b)\left(b_y\partial_x^2 + (a_x+b_y)\partial_x\partial_y\right) + \\ &\quad 3\exp(2a+b)\left((a_y+b_y)\partial_x\partial_y + a_x\partial_y^2\right) + \\ &\quad \exp(a+2b)\left(2b_{xy} + 3b_x b_y + a_x b_y\right)\partial_x + \\ &\quad \exp(2a+b)\left(b_{yy} + b_y^2 + a_y b_y\right)\partial_x + \\ &\quad \exp(a+2b)\left(a_{xx} + a_x^2 + a_x b_x\right)\partial_y + \\ &\quad \exp(2a+b)\left(2a_{xy} + a_x b_y + 3a_x a_y\right)\partial_y .\end{aligned}$$

- Quantization, $k = 2$.

$$\begin{aligned}\widehat{\sigma_2} &= a_{11}\partial_1^2 + 2a_{12}\partial_1\partial_2 + a_{22}\partial_2^2 + \\ &\quad a_{11}\left(b_x\partial_x\right) + a_{12}\left(b_y\partial_x + a_x\partial_y\right) + a_{22}\left(a_y\partial_y\right).\end{aligned}$$

Invariants

- Wagner metric: σ is a homogeneous polynomial of degree 3 on the plane V. Then

$$W(\sigma) = \sqrt[3]{\mathrm{Hess}_2(\sigma)}\,\mathrm{Hess}(\sigma)^{-1}$$

  is Wagner metric, invariantly associated with σ. (Definite in the hyperbolic and indefinite for the ultrahyperbolic case)

- In our case

$$\frac{D_{\sigma}^{2/3}}{4} W(\sigma) = \left(a_{112}a_3 - a_{122}^2\right) dx^2 + \left(a_{112}a_{122} - a_{111}a_3\right) dx \cdot dy + \left(a_{111}a_{122} - a_{112}^2\right) dy^2.$$

- Torsion form: $\theta_h = a_x dx + b_y dy.$
- Invariant coframe: $\left\langle \theta_h, \theta_h^{\perp} \right\rangle$
- Invariants: coefficients of tensors $\sigma_3(A), \sigma_2(A), \sigma_2(A), \sigma_0(A)$ in the invariant coframe and their invariant derivatives.