Symmetry gaps for geometric structures (Lecture 5)

Dennis The

Department of Mathematics & Statistics
UiT The Arctic University of Norway

(based on joint work with Boris Kruglikov)

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Lecture 5 Outline

1. From geometry to algebra: proof outline
2. Realizability: proof outline
3. Dynkin diagram recipes

From geometry to algebra

Among reg/nor geometries of type (G, P), submax sym dim is

$$\mathfrak{S} = \max\{\dim(\mathfrak{inf}(\mathcal{G}, \omega)) : \kappa_H \not\equiv 0\}.$$

Define $\mathfrak{a}^\phi := \mathrm{pr}_\mathfrak{g}(\mathfrak{g}_-, \mathfrak{ann}(\phi))$.

Theorem (Kruglikov, T. (2013))

Let $(\mathcal{G} \to M, \omega)$ be any regular, normal G/P geometry. Then $\dim(\mathfrak{inf}(\mathcal{G}, \omega)) \leq \dim(\mathfrak{a}^{\kappa_H(u)})$, $\forall u$ in some open dense subset of $\mathcal{G}$.

For a geometry realizing $\mathfrak{S}$, we have $\kappa_H(u) \neq 0$ for some $u \in \mathcal{G}$, so by continuity also on a nbd of u. $\therefore$ WLOG, above inequality holds.

Define $\mathfrak{U} := \max\{\dim(\mathfrak{a}^\phi) \mid 0 \neq \phi \in H^2_+(\mathfrak{g}_-, \mathfrak{g})\}$.

Corollary (Universal upper bound)

$\mathfrak{S} \leq \mathfrak{U} < \dim(\mathfrak{g})$.

Proof outline of Thm

Čap–Neusser (2009):

- Fix any $u \in \mathcal{G}$. Then $\omega_u : \mathfrak{inf}(\mathcal{G}, \omega) \hookrightarrow \mathfrak{g}$ (linearly).
- Bracket on $\mathfrak{f} = \mathrm{im}(\omega_u)$ is $[X, Y]_{\mathfrak{f}} := [X, Y]_{\mathfrak{g}} - \kappa_u(X, Y)$.
- Regularity: $\mathfrak{f}$ is filtered, so $\mathfrak{s} = gr(\mathfrak{f}) \subset \mathfrak{g}$ is a graded subalg.
- $\mathfrak{s}_0 \subset \mathfrak{ann}(\kappa_H(u))$. [$\Rightarrow$ uniqueness of max. sym. model.]

$(*)$: $\boxed{[\mathfrak{s}_{i+1}, \mathfrak{g}_{-1}] \subset \mathfrak{s}_i}$ $(i \geq -1)$ $\quad \Rightarrow \quad$ $\mathfrak{s} \subset \mathrm{pr}_{\mathfrak{g}}(\mathfrak{g}_-, \mathfrak{s}_0) \subset \mathfrak{a}^{\kappa_H(u)}$,
so $\dim(\mathfrak{s}) \leq \mathfrak{U}$ when $\kappa_H(u) \neq 0$.

BUT: "Tanaka property" $(*)$ isn't always true!

Definition

$x \in M$ is a regular point iff $\forall i$, $\dim(\mathfrak{s}_i)$ is loc. constant near x.

Proof outline.

(1) Prop: At regular points, $(*)$ is true. (Proof on next slides.)
(2) Lemma: The set of regular points is open and dense in M.
(3) Any nbd of a non-flat point contains a non-flat regular pt. □

Proof: The Tanaka property - 1

Let $\mathcal{S} := \mathfrak{inf}(\mathcal{G}, \omega)$. Fix $u \in \pi^{-1}(x)$. The filtration $\{\mathfrak{g}^i\}$ on $\mathfrak{g}$ gives a filtration of $\mathcal{S}$, i.e. $\mathcal{S}^j := \{\xi \in \mathcal{S} \mid \omega_u(\xi) \in \mathfrak{g}^j\}$. Let $\mathfrak{f}^j := \omega_u(\mathcal{S}^j)$.

WTS: $\forall i \geq -1$, $[\mathfrak{s}_{i+1}, \mathfrak{g}_{-1}] \subset \mathfrak{s}_i$, i.e. $[\mathfrak{f}^{i+1}, \mathfrak{g}^{-1}] \subset \mathfrak{f}^i + \mathfrak{g}^{i+1}$.

1. Let $X = \omega_u(\xi) \in \mathfrak{f}^{i+1} \subset \mathfrak{p}$ and $Y = \omega_u(\eta) \in \mathfrak{g}^{-1}$ in terms of $\xi \in \mathcal{S}^{i+1}$ and $\eta \in \mathfrak{X}(\mathcal{G})^P$. Claim: $[X, Y] = -\omega_u([\xi, \eta]) \in \mathfrak{g}^i$.
   - Since ξ is a symmetry, then

$$\begin{aligned} 0 &= (\mathcal{L}_\xi \omega)(\eta) = d\omega(\xi, \eta) + d(\omega(\xi))(\eta) \\ &= \xi(\omega(\eta)) - \omega([\xi, \eta]). \end{aligned}$$

   - Since $X \in \mathfrak{p}$ and $\omega(\zeta_X) = X$, then $\xi_u = (\zeta_X)_u$.
   - $\omega(\eta)$ is P-equiv, so $\mathcal{L}_{\zeta_X}(\omega(\eta)) = -\mathrm{ad}_X(\omega(\eta))$
   - Finally,

$$\begin{aligned} \omega_u([\xi, \eta]) &= \xi_u(\omega(\eta)) = (\zeta_X)_u(\omega(\eta)) \\ &= -\mathrm{ad}_X(\omega_u(\eta)) = -[X, Y] \in \mathfrak{g}^i. \end{aligned}$$

Proof: The Tanaka property - 2

For $X = \omega_u(\xi) \in \mathfrak{f}^{i+1}$ and $Y = \omega_u(\eta) \in \mathfrak{g}^{-1}$, we showed $[X, Y] = -\omega_u([\xi, \eta]) \in \mathfrak{g}^i$. WTS: $[X, Y] \in \mathfrak{f}^i + \mathfrak{g}^{i+1}$.

2. $\mathcal{G} = \mathcal{G}_\nu \to \ldots \to \mathcal{G}_0 \to M$ with $\mathcal{G}_i = \mathcal{G}/P_+^{i+1} \xrightarrow{\pi_i} M$. Then $\mathcal{S}$ projects isomorphically to $\mathcal{S}^{(i)} \subset \mathfrak{X}(\mathcal{G}_i)$.
   - x regular pt $\Rightarrow \mathcal{S}^{(i)}$ is a constant rank & involutive distribution. By Frobenius, $\exists$ fcns $\{F_j\}$ on $\mathcal{G}_i$; level sets foliate by integral submflds of $\mathcal{S}^{(i)}$. Thus, $\xi^{(i)} \cdot F_j = 0$, $\forall j$.
   - Since $\xi \in \mathcal{S}^{i+1}$, then $\forall u_i \in \pi_i^{-1}(x)$, $\xi^{(i)}_{u_i} = 0$. Hence,

$$
\begin{aligned}
[\xi^{(i)}, \eta^{(i)}]_{u_i} \cdot F_j &= \xi^{(i)}_{u_i}(\eta^{(i)} \cdot F_j) - \eta^{(i)}_{u_i}(\xi^{(i)} \cdot F_j) = 0. \\
\Rightarrow \quad & [\xi^{(i)}, \eta^{(i)}]_{u_i} = (\xi')^{(i)}_{u_i}, \quad \text{some } \xi' \in \mathcal{S}. \\
\Rightarrow \quad & \omega_u([\xi, \eta]) = \omega_u(\xi') + \mathfrak{g}^{i+1}, \quad \text{some } \xi' \in \mathcal{S}.
\end{aligned}
$$

Conclusion: $[X, Y] \in \mathfrak{f}^i + \mathfrak{g}^{i+1}$. □

Realizability: proof outline

Define $\mathfrak{f} = \mathfrak{a} := \mathfrak{a}^{\phi_0}$ as *vector spaces*, but with deformed bracket

$$[X, Y]_{\mathfrak{f}} := [X, Y]_{\mathfrak{a}} - \phi_0(X, Y).$$

Kostant gives *explicit* l.w. $\phi_0 \in \mathbb{V} \subset H^2 \cong \ker(\Box) \subset \bigwedge^2 \mathfrak{g}_-^* \otimes \mathfrak{g}$.
Namely, for $w = (jk) \in W^{\mathfrak{p}}(2)$, have $\phi_0 = e_{\alpha_j} \wedge e_{s_{\alpha_j}(\alpha_k)} \otimes e_{w(-\lambda)}$.

Q: Is this even a Lie algebra? To have hope, we need $\boxed{e_{w(-\lambda)} \in \mathfrak{a}}$.

Proposition

If $w(-\lambda) \in \Delta^-$ above, then $\mathfrak{f}$ is a filtered Lie algebra.

Lemma

If $\mathfrak{g}$ is simple, $w \in W^{\mathfrak{p}}(2)$, and $w(-\lambda) \in \Delta^+$, then $\mathrm{rank}(\mathfrak{g}) = 2$.

Non-exceptions: $\mathfrak{f}/\mathfrak{f}^0 \rightsquigarrow$ non-flat model, $\dim(\mathfrak{f}) = \mathfrak{U}$, so $\mathfrak{S} = \mathfrak{U}$.

Have algorithm for constructing an explicit submax. sym. model.

Dynkin diagram recipes - 1

Proposition (Extremal vectors maximize $\mathfrak{a}^\phi := \mathrm{pr}_\mathfrak{g}(\mathfrak{g}_-, \mathfrak{ann}(\phi))$)

Over $\mathbb{C}$, *if* $\mathbb{V}$ *is a* $\mathfrak{g}_0$*-irrep,* $\phi_0 \in \mathbb{V}$ *is extremal, then* $\forall \phi \in \mathbb{V}\backslash\{0\}$,

$$\dim(\mathfrak{a}_k^\phi) \leq \dim(\mathfrak{a}_k^{\phi_0}), \quad \forall k \geq 0.$$

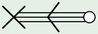
Thus, to find upper bound $\mathfrak{U}$, it suffices to evaluate $\dim(\mathfrak{a}^{\phi_0})$ (on each irrep $\mathbb{V} \subset H^2_+(\mathfrak{g}_-, \mathfrak{g})$). This leads to various DD recipes:

1. $\mathfrak{g} = \mathfrak{g}_- \oplus \overbrace{\mathfrak{g}_0 \oplus \mathfrak{g}_+}^{\mathfrak{p}}$, and $\mathfrak{g}_0 = \mathfrak{z}(\mathfrak{g}_0) \oplus \mathfrak{g}_0^{ss}$ with

   $$\begin{cases} \dim(\mathfrak{z}(\mathfrak{g}_0)) = \#\text{ crosses;} \\ \mathfrak{g}_0^{ss}\text{ DD} \to \text{remove crosses.} \end{cases}$$

   Since $\dim(\mathfrak{g}_-) = \dim(\mathfrak{g}_+)$, get $n = \dim(\mathfrak{g}/\mathfrak{p})$ and $\dim(\mathfrak{p})$.

Example (G_2/P_1)

 , $\dim(\mathfrak{g}_0) = 4, \qquad n = 5.$

Dynkin diagram recipes - 2

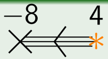
Let $\mathbb{V} \subset H^2_+$ be a $\mathfrak{g}_0$-irrep and $\phi_0 \in \mathbb{V}$ a l.w. vector.

2. $\mathfrak{q} := \{X \in \mathfrak{g}_0^{ss} \mid X \cdot \phi_0 = \lambda \phi_0\}$ is parabolic, and

$$\boxed{\dim(\mathfrak{ann}(\phi_0)) = (\#\text{crosses}) - 1 + \dim(\mathfrak{q})}$$

D.D. Notation: If $\neq 0$ on uncrossed node, put $*$. This describes a parabolic in $\mathfrak{g}_0^{ss}$, whose dim we know how to compute.

Example (G_2/P_1)

$H^2_+ =$ 

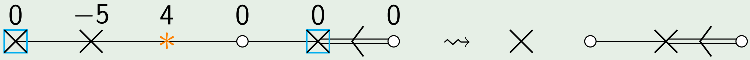
, $\quad \dim(\mathfrak{ann}(\phi_0)) = 2.$

Dynkin diagram recipes - 3

D.D. Notation: If 0 over $\times \rightsquigarrow$ put $\square$.

3. Remove all $*$ and $\times$, except $\square$ (also remove adj. edges). Then remove connected components w/o $\square$. Obtain $(\bar{\mathfrak{g}}, \bar{\mathfrak{p}})$.

Example



Proposition (Prolongation criterion)

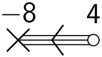
No $\square \Leftrightarrow \dim(\mathfrak{a}_+^{\phi_0}) = 0$. *Otw,* $\dim(\mathfrak{a}_+^{\phi_0}) = \dim(\bar{\mathfrak{g}}/\bar{\mathfrak{p}})$.

Proposition (Maximal parabolics)

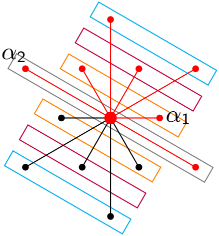
Single cross $\Rightarrow$ *no* $\square$, *so* $\mathfrak{a}_+^{\phi_0} = 0$.

Example: G_2/P_1

Let's be more concrete and prove that $\mathfrak{a}_+^{\phi_0} = 0$ for G_2/P_1.

Recall $H^2_+(\mathfrak{g}_-, \mathfrak{g}) =$ 

has l.w. $8\lambda_1 - 4\lambda_2 = 4\alpha_1$.



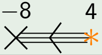
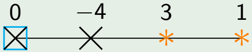
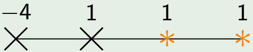
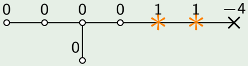
$$\begin{cases} \mathfrak{g}_{-1} = \mathrm{span}\{e_{-\alpha_1}, e_{-\alpha_1-\alpha_2}\} \\ \mathfrak{a}_0 = \ker\{\alpha_1\} \oplus \mathfrak{g}_{-\alpha_2} \\ \mathfrak{a}_1 = ? \end{cases}$$

Let $\mathrm{E} = ae_{\alpha_1} + be_{\alpha_1+\alpha_2} \in \mathfrak{g}_1$. Then

- $[\mathrm{E}, e_{-\alpha_1}] \in \mathfrak{a}_0$ forces $b = 0$.
- Take $\mathrm{E} = e_{\alpha_1}$ and $\mathrm{F} = e_{-\alpha_1}$, so $[\mathrm{E},\mathrm{F}] = \mathrm{H} := h_{\alpha_1} \in \mathfrak{h}$.
- But $[\mathrm{H}, e_{-\alpha_1}] = -2e_{-\alpha_1}$, so $\mathrm{H} \notin \ker\{\alpha_1\}$. Thus, $\mathfrak{a}_1 = 0$, hence $\mathfrak{a}_+ = 0$.

More examples

Example

G/P	H^2_+ components	n	$\dim(\mathfrak{a}_0^{\phi_0})$	$\dim(\mathfrak{a}_+^{\phi_0})$	$\dim(\mathfrak{a}^{\phi_0})$
G_2/P_1		5	2	0	7
$A_4/P_{1,2}$		7	6	1	14
		7	6	0	13
E_8/P_8		57	90	0	147

Summary & open questions

- We motivated Cartan geometry as an "upstairs" solution to Cartan's method of equivalence for some type of "downstairs" geometric structure. This gives a notion of curvature.
- Tanaka prolongation bounds the sym dim of a filtered G_0-structure.
- Parabolic setting: used Kostant's theorem to efficiently compute cohomology groups, which are useful for Tanaka prolongation and harmonic curvature.
- Symmetry gaps: we passed from the geometric to the algebraic setting and got a universal upper bound.

Open questions:

- Gaps for other (real) structures? (Kruglikov, Winther,...)
- Classification of *all* submaximally symmetric models?
- Non-parabolic geometries, e.g. Higher order ODE (systems)?
- Dim of submax space of solns of almost-Einstein scales, Killing tensors, etc. (more generally, of BGG operators)?