# Symmetry gaps for geometric structures (Lecture 4)

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(based on joint work with Boris Kruglikov)

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# Recap

Formulation of the symmetry gap problem "upstairs":

> Among all regular, normal $(\mathcal{G} \to M, \omega)$ of (fixed) type $(G, P)$, calculate the submax sym dim
>
> $$\mathfrak{S} := \max\{\dim(\mathfrak{inf}(\mathcal{G}, \omega)) \mid \kappa_H \not\equiv 0\}.$$

- $\kappa_H$ is valued in $H^2_+(\mathfrak{g}_-, \mathfrak{g})$.
- $H^1_{\geq 0}(\mathfrak{g}_-, \mathfrak{g}) = 0$ iff $\mathrm{pr}(\mathfrak{g}_-, \mathfrak{g}_0) = \mathfrak{g}$.

GOAL: Formulate Kostant's theorem, use it to compute $H^1$ and $H^2$, and interpret the results.

# Lecture 4 Outline

1. Root diagrams, Dynkin diagrams, and the Weyl group
2. Lie algebra cohomology & Kostant's theorem

# Root diagrams & Dynkin diagrams

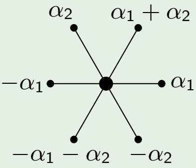
$\mathfrak{g}$ $\mathbb{C}$-s.s., $\Delta$ root system, Killing form: $B(x, y) = \mathrm{tr}(\mathrm{ad}_x \circ \mathrm{ad}_y)$.

**Theorem**

*$B|_{\mathfrak{g}}$ & $B|_{\mathfrak{h}}$ are ndg, get induced pos.def $\langle \cdot, \cdot \rangle$ on $V = \mathrm{span}_{\mathbb{R}} \Delta \subset \mathfrak{h}^*$.*
*(Can compute angles btw roots and relative lengths!)*

Cartan matrix: $c_{ij} = \langle \alpha_i, \alpha_j^\vee \rangle$, where $\alpha^\vee = \frac{2\alpha}{\langle \alpha, \alpha \rangle}$. (Note $c_{ii} = 2$.)
FACT: For $i \neq j$, $c_{ij} \in \mathbb{Z}_{\leq 0}$ and $c_{ij} c_{ji} \in \{0, 1, 2, 3\}$.
Dynkin diagram: Graph with $\alpha_i \leftrightarrow$ node $i$; bond from $i$ to $j$ of multiplicity $c_{ij} c_{ji}$, directed towards shorter root if $c_{ij} c_{ji} > 1$.
Serre relations: can recover $\mathfrak{g}$ from DD.

**Example ($\mathfrak{g} = \mathfrak{sl}_3$)**

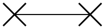
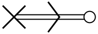
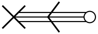
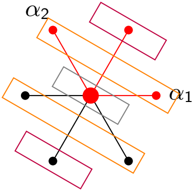
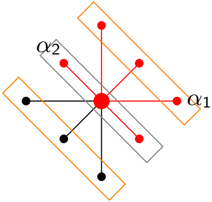
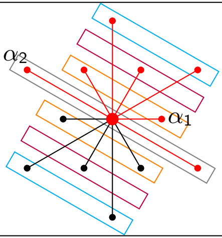



$$c_{ij} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

# Gradings revisited

Encode $\mathfrak{p}_I$ (or $Z = \sum_{i \in I} Z_i$) by putting crosses $\forall i \in I$ in the DD.

FACT: Parabolics in $\mathbb{C}$-s.s. $\mathfrak{g}$ are classified by such crossed DD.

| $A_2 := \mathfrak{sl}_3$ | $B_2 := \mathfrak{so}_{2,3}$ | $G_2$ |
|---|---|---|
|  |  |  |
|  |  |  |
| 2nd order ODE | 3-dim Lorentzian conformal | $(2,3,5)$-dist. |

$\mathfrak{g}_0 = \mathfrak{z}(\mathfrak{g}_0) \oplus \mathfrak{g}_0^{ss}$. Recipe: $\begin{cases} \dim(\mathfrak{z}(\mathfrak{g}_0)) = \#\text{crosses}; \\ \mathfrak{g}_0^{ss} \leftrightarrow \text{DD after omitting crosses.} \end{cases}$

Simple rts $\{\alpha_i\}$. Fundamental weights $\{\lambda_i\}$ def. by $\langle \lambda_i, \alpha_j^\vee \rangle = \delta_{ij}$.
Wrt Cartan matrix $c_{ij}$ and its inverse $c^{ij}$, we have the basis change:

$$\alpha_i = c_{ij}\lambda_j, \quad \lambda_i = c^{ij}\alpha_j.$$

**Example (Highest root)**

$\mathfrak{sl}_3$: $\alpha_1 + \alpha_2 = \lambda_1 + \lambda_2$; $\quad G_2$: $3\alpha_1 + 2\alpha_2 = \lambda_2$.

**Theorem (Highest weight)**

*Given $\mathfrak{g}$ $\mathbb{C}$-s.s., there is a 1-1 correspondence between $\mathfrak{g}$-irreps and dominant integral weights $\lambda = \sum_i r_i \lambda_i$, where $r_i \in \mathbb{Z}_{\geq 0}$.*

Encode these numbers on a Dynkin diagram, e.g. $\overset{2}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{3}{\circ}$ .

**Example ($\mathfrak{sl}_2$-irreps)**

Std. rep $\mathbb{C}^2$: $\overset{1}{\circ}$ ; adj. rep: $\mathfrak{sl}_2 \cong S^2\mathbb{C}^2$: $\overset{2}{\circ}$ ; $S^k\mathbb{C}^2$: $\overset{k}{\circ}$

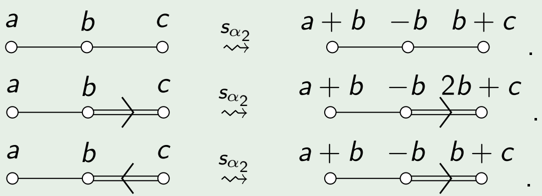
# Weyl group

For any $\alpha \in \Delta$, have reflection $s_\alpha(\lambda) = \lambda - \frac{2\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} \alpha$.

Weyl group: $W = \langle s_\alpha : \alpha \in \Delta \rangle \leq O(V)$. Have $\Delta$ is $W$-invariant. Let $\{\alpha_i\}$ be simple roots for $\Delta$. FACT: $W$ is generated by $s_{\alpha_i}$, so any $w \in W$ is a word, e.g. $(12) = s_{\alpha_1} \circ s_{\alpha_2}$.

Q: How does $s_{\alpha_j}$ act on $\lambda = \sum_i r_i \lambda_i$?
A: Let $c = r_j$. Add $c$ to adjacent coeffs in DD, with multiplicity if $\exists$ multiple bond directed to the adjacent node. Replace $c$ by $-c$.

## Example




Affine $W$-action: $w \bullet \lambda = w(\lambda + \rho) - \rho$, where $\rho = \sum_i \lambda_i$.

# Lie algebra cohomology $H^k(\mathfrak{g}_-, \mathfrak{g})$

Let $C^k(\mathfrak{g}_-, \mathfrak{g}) = \bigwedge^k (\mathfrak{g}_-)^* \otimes \mathfrak{g}$. We have differentials

$$0 \to C^0(\mathfrak{g}_-, \mathfrak{g}) \stackrel{\partial}{\to} C^1(\mathfrak{g}_-, \mathfrak{g}) \stackrel{\partial}{\to} C^2(\mathfrak{g}_-, \mathfrak{g}) \stackrel{\partial}{\to} \dots$$

(Formulas for $\partial$ on $C^0$ and $C^1$ later.) Define

$$H^k(\mathfrak{g}_-, \mathfrak{g}) = \frac{\ker(\partial : C^k \to C^{k+1})}{\mathrm{im}(\partial : C^{k-1} \to C^k)}.$$

Since $\mathfrak{g}_-, \mathfrak{g}$ are $\mathfrak{g}_0$-modules, then so are $C^k(\mathfrak{g}_-, \mathfrak{g})$ and $H^k(\mathfrak{g}_-, \mathfrak{g})$.

# Kostant's theorem

Let $\mathfrak{g}$ be $\mathbb{C}$-simple with highest weight $\lambda$, and $\mathfrak{p} \subset \mathfrak{g}$ parabolic. Let $\mathbb{V}_\mu = \mathfrak{g}_0$-irrep with lowest weight $\mu$. Simplified version of Kostant's thm:

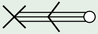
**Theorem**

*As $\mathfrak{g}_0$-modules,* $$H^k(\mathfrak{g}_-, \mathfrak{g}) = \bigoplus_{w \in W^{\mathfrak{p}}(k)} \mathbb{V}_{-w\bullet\lambda}.$$

Here, $W^{\mathfrak{p}}(k)$ = length $k$ words of the Hasse subset of $W$. Instead of defining $W^{\mathfrak{p}}$ in general, here it is for $k = 1$ and $k = 2$:

- $W^{\mathfrak{p}}(1) = \{(i) : \text{node } i \text{ in DD is crossed}\}$
- $W^{\mathfrak{p}}(2) = \{(jk) : j \text{ is crossed; } k \text{ is crossed or adjacent to } j\}$.

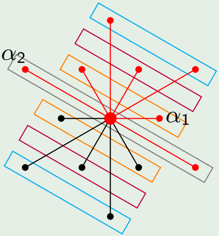
**Example ($G_2/P_1$)**

 $\Rightarrow$ $W^{\mathfrak{p}}(1) = \{(1)\}$, $W^{\mathfrak{p}}(2) = \{(12)\}$.

# $H^0(\mathfrak{g}_-, \mathfrak{g})$ example

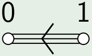
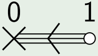
$\partial^0 : C^0 \to C^1$ is $(\partial^0 v)(x) = [x, v]$. So, $v \in \ker(\partial^0)$ iff $\mathfrak{g}_- \cdot v = 0$.

**Example ($G_2/P_1$)**




$\Rightarrow$ $H^0(\mathfrak{g}_-, \mathfrak{g}) = \ker(\partial) = \mathfrak{g}_{-3}$.

This is overkill in this case, but let's check it via Kostant:

1. $\mathfrak{g}$ has h.w. $\lambda_2 =$   ; $W^{\mathfrak{p}}(0) = \{\mathrm{id}\}$.
2. $H^0(\mathfrak{g}_-, \mathfrak{g}) \overset{(*)}{=}$   $\cong \mathfrak{g}_{-3}$, i.e. $\mathfrak{g}_0$-module with l.w. $-\lambda_2 = -3\alpha_1 - 2\alpha_2$.

($(*)$: Use "minus lowest weight" convention here and below.)

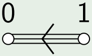
# $H^1(\mathfrak{g}_-, \mathfrak{g})$ example

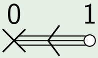
$\partial^1 : C^1 \to C^2$ is $(\partial\eta)(x,y) = [x, \eta(y)] - [y, \eta(x)] - \eta([x,y])$.

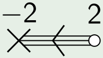
$$H^1(\mathfrak{g}_-, \mathfrak{g}) = \frac{\ker(\partial^1)}{\mathrm{im}(\partial^0)} = \frac{\mathfrak{g}\text{-valued derivations on } \mathfrak{g}_-}{\langle \mathrm{ad}_v|_{\mathfrak{g}_-} : v \in \mathfrak{g}\rangle}$$

Thus, $H^1_{\geq 0}(\mathfrak{g}_-, \mathfrak{g}) = 0$ iff $\mathrm{pr}(\mathfrak{g}_-) = \mathfrak{g}$.

## Example ($G_2/P_1$)

1. $G_2$ h.w. $\lambda_2 =$ 

; $c_{ij} = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$; $W^{\mathfrak{p}}(1) = \{(1)\}$.
2. $(1) \bullet ($ 

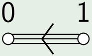
$) =$ 

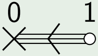
$\overset{\text{Kostant}}{=} H^1(\mathfrak{g}_-, \mathfrak{g})$
(3-dim module $S^2\mathbb{C}^2$ for $\mathfrak{g}_0^{ss} \cong \mathfrak{sl}_2$).
3. L.w. $2\lambda_1 - 2\lambda_2 = -2\alpha_1 - 2\alpha_2$ with $Z_1$-grading $= -2$.
4. Conclusion: $H^1_{\geq 0}(\mathfrak{g}_-, \mathfrak{g}) = 0$, so $\mathrm{pr}(\mathfrak{g}_-) = \mathfrak{g}$, i.e. $(2,3,5)$-dist. have at most 14-dim symmetry.

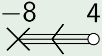
# $H^2(\mathfrak{g}_-, \mathfrak{g})$ example

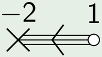
Recall: $\kappa_H$ is valued in $H^2_+(\mathfrak{g}_-, \mathfrak{g})$.

## Example ($G_2/P_1$)

1. $\mathfrak{g}$ has h.w. $\lambda_2 =$ 

; $W^{\mathfrak{p}}(2) = \{(12)\}$.
2. $(12) \bullet ($ 

$) =$ 

$\overset{\text{Kostant}}{=} H^2(\mathfrak{g}_-, \mathfrak{g})$
(5-dim module $S^4\mathbb{C}^2$ for $\mathfrak{g}_0^{ss} \cong \mathfrak{sl}_2$).
3. L.w. $8\lambda_1 - 4\lambda_2 = 4\alpha_1$ with $Z_1$-grading $= +4$.
4. $\mathfrak{g}_1 \cong$ 

$\cong (\mathfrak{g}_{-1})^*$, so $H^2_+(\mathfrak{g}_-, \mathfrak{g}) \cong S^4(\mathfrak{g}_{-1})^*$.
This identifies the existence of Cartan's binary quartic field for $(2,3,5)$-distributions.