# Symmetry gaps for geometric structures (Lecture 3)

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(based on joint work with Boris Kruglikov)

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Sophus Lie Center in Nordfjordeid

# Lecture 3 Outline

1. Parabolic geometry:
   - parabolic subalgebras of semisimple Lie algebras
   - normalization conditions ("regular, normal")
   - fundamental invariant $\kappa_H$
2. Statement of symmetry gap results.

# Fundamental theorem of parabolic geometries

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What does "parabolic" mean? ("regular, normal" $\to$ later)

# Parabolic subalgebras

**Definition**

*Let* $\mathfrak{g}$ *be semisimple. A subalgebra* $\mathfrak{p} \subset \mathfrak{g}$ *is* *parabolic* *if* $\mathfrak{p} = \mathfrak{g}_{\geq 0}$ *for some* $\mathbb{Z}$*-grading* $\mathfrak{g} = \bigoplus_i \mathfrak{g}_i$ *(with* $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$, $\forall i, j$*).*

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**Example ($\mathfrak{g} = \mathfrak{sl}_3$, $\mathfrak{p} = \mathfrak{p}_{1,2} =$ (trace-free) upper triangular)**

$$\mathfrak{p} = \begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & 1 \\ -2 & -1 & 0 \end{pmatrix} \subset \mathfrak{g}. \quad \text{We have } \mathfrak{g} = \overbrace{\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}}^{\text{Heisenberg}} \oplus \overbrace{\mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2}^{\mathfrak{p}}.$$

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The above $\mathfrak{p}$ is the Borel subalgebra $\mathfrak{b}$. (Precise defn shortly.)
Alternative: A subalg is parabolic if it contains a Borel.

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Let $\mathfrak{g}$ be $\mathbb{C}$-semisimple, $\mathfrak{h} \subset \mathfrak{g}$ Cartan subalg.

**Example ($\mathfrak{g} = \mathfrak{sl}_2$)**

Let $\mathsf{H} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\mathsf{E} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\mathsf{F} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Commutators:

$$[\mathsf{H}, \mathsf{E}] = +2\mathsf{E}, \quad [\mathsf{H}, \mathsf{F}] = -2\mathsf{F}, \quad [\mathsf{E}, \mathsf{F}] = \mathsf{H}.$$

Eigenvalues of $\mathrm{ad}_{\mathsf{H}}$ wrt $(\mathsf{F}, \mathsf{H}, \mathsf{E})$-basis are $-2, 0, +2$. Here $\mathfrak{h} = \langle \mathsf{H} \rangle$.

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- $\forall \alpha \in \Delta$: $\dim(\mathfrak{g}_\alpha) = 1$ and $-\alpha \in \Delta$.
- $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$. (Have equality if $\alpha, \beta, \alpha + \beta \in \Delta$.)

**Example (Root space decomp: $\mathfrak{g} = \mathfrak{sl}_3$, $\mathfrak{h}$ = trace-free diagonal)**

Let $\epsilon_i(\text{diag}(a_1, a_2, a_3)) = a_i$. Simple rts: $\alpha_1 = \epsilon_1 - \epsilon_2$, $\alpha_2 = \epsilon_2 - \epsilon_3$.

| $\alpha \in \Delta$ | $\alpha_1$ | $\alpha_2$ | $\alpha_1 + \alpha_2$ | $-\alpha_1$ | $-\alpha_2$ | $-\alpha_1 - \alpha_2$ |
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| Basis for $\mathfrak{g}_\alpha$ | $E_{12}$ | $E_{23}$ | $E_{13}$ | $E_{21}$ | $E_{32}$ | $E_{31}$ |

# Lie algebra gradings

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Simple roots $\{\alpha_i\} \subset \mathfrak{h}^*$, dual basis $\{Z_i\} \subset \mathfrak{h}$, i.e. $\alpha_i(Z_j) = \delta_{ij}$
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**Example (Two parabolics in $\mathfrak{sl}_3$)**

$$Z_1 + Z_2 \rightsquigarrow \mathfrak{b} = \mathfrak{p}_{1,2} = \begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & 1 \\ -2 & -1 & 0 \end{pmatrix}, \quad Z_1 \rightsquigarrow \mathfrak{p}_1 = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

# Root diagrams & Dynkin diagrams

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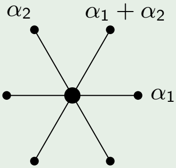


$$c_{ij} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

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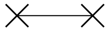
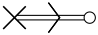
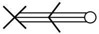
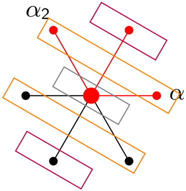
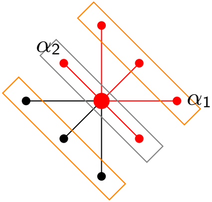
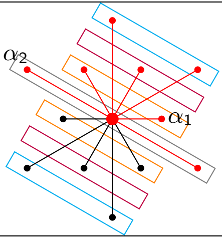
| $A_2 := \mathfrak{sl}_3$ | $B_2 := \mathfrak{so}_{2,3}$ | $G_2$ |
|---|---|---|
|  |  |  |
|   |   |   |
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# Gradings revisited

Encode $\mathfrak{p}_I$ (or $Z = \sum_{i \in I} Z_i$) by putting crosses $\forall i \in I$ in the DD.
FACT: Parabolics in $\mathbb{C}$-s.s. $\mathfrak{g}$ are classified by such crossed DD.

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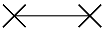
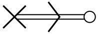
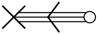
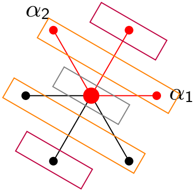
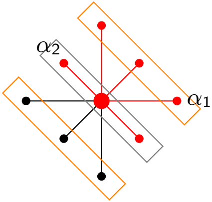
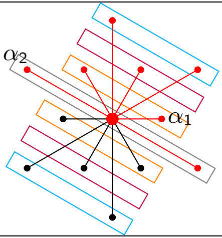
|---|---|---|
|  |  |  |
|  |  |  |
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In these cases, we immediately conclude $\mathfrak{g} \hookrightarrow \text{pr}(\mathfrak{m}, \mathfrak{g}_0)$. (For 3rd case, use $\text{pr}(\mathfrak{m})$.) In fact, $\mathfrak{g} \cong \text{pr}(\mathfrak{m}, \mathfrak{g}_0)$; see Lecture 4.

# Regular & normal

$(\mathfrak{g}, \mathfrak{p}) \rightsquigarrow \mathfrak{g} = \mathfrak{g}_- \oplus \overbrace{\mathfrak{g}_0 \oplus \mathfrak{g}_+}^{\mathfrak{p}}$. Have $\mathfrak{p}$-inv. filtrands $\mathfrak{g}^i = \bigoplus_{j \geq i} \mathfrak{g}_j$.

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1. *regular if $\kappa(\mathfrak{g}^i, \mathfrak{g}^j) \subset \mathfrak{g}^{i+j+1}$, $\forall i, j$. ($\iff$ $\kappa$ is valued in the subspace of $\bigwedge^2 (\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$ on which $Z$ has pos. eigenvalues.)*

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2. *normal if $\partial^* \kappa = 0$, where $\partial^* : \bigwedge^2 \mathfrak{g}_+ \otimes \mathfrak{g} \to \mathfrak{g}_+ \otimes \mathfrak{g}$ is the Lie algebra homology differential defined by*

$$\partial^*(X \wedge Y \otimes v) = Y \otimes (X \cdot v) - X \otimes (Y \cdot v) + [X,Y] \otimes v.$$

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$$\rightsquigarrow \bigwedge^2(\mathfrak{g}_-)^* \otimes \mathfrak{g} = \overbrace{\mathrm{im}(\partial^*) \oplus \ker(\Box)}^{\ker(\partial^*)} \oplus \mathrm{im}(\partial), \qquad \ker(\Box) \cong H^2(\mathfrak{g}_-, \mathfrak{g})$$

where $\ker(\Box) \oplus \mathrm{im}(\partial) = \ker(\partial)$.

cohomology!

# Harmonic curvature

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**Examples (Harmonic curvature)**

- conformal geometry: Weyl ($n \geq 4$) or Cotton ($n = 3$);
- $(2, 3, 5)$-distributions: binary quartic.
- 2nd order ODE: Tresse invariants $I_1, I_2$.

# Summary

The (locally) flat model is the *unique* max. sym. model, so we can now precisely formulate the symmetry gap problem "upstairs":

Among all regular, normal $(\mathcal{G} \to M, \omega)$ of (fixed) type $(G, P)$, calculate the submax sym dim

$$\mathfrak{S} := \max\{\dim(\mathfrak{inf}(\mathcal{G}, \omega)) \mid \kappa_H \not\equiv 0\}.$$

- $\kappa_H$ is valued in $H^2_+(\mathfrak{g}_-, \mathfrak{g})$.
- $H^1_{\geq 0}(\mathfrak{g}_-, \mathfrak{g})$ is relevant for calculating $\mathrm{pr}(\mathfrak{m}, \mathfrak{g}_0)$.

*Kostant (1961), Baston–Eastwood (1989)*: Dynkin diagram algorithm to calculate $H^2_+(\mathfrak{g}_-, \mathfrak{g})$ as a $\mathfrak{g}_0$-module.

# Kostant's theorem - sample output

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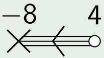
**Example ((2, 3, 5)-distributions: $G_2/P_1$ geometry)**

As a $\mathfrak{g}_0 = \mathfrak{gl}_2(\mathbb{R})$ module,

$$H^2_+(\mathfrak{g}_-, \mathfrak{g}) =$$




$$= \bigodot^4 (\mathbb{R}^2)^*,$$

i.e. binary quartic, c.f. Cartan (1910).

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$$\begin{aligned} \mathrm{pr}_{\mathfrak{g}}(\mathfrak{g}_-, \mathfrak{a}_0) &:= \mathfrak{g}_- \oplus \mathfrak{a}_0 \oplus \mathfrak{a}_+ \\ \mathfrak{a}_i &:= \{X \in \mathfrak{g}_i \mid [X, \mathfrak{g}_{-1}] \subset \mathfrak{a}_{i-1}\} \quad (i \geq 1). \end{aligned}$$

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*Let $(\mathcal{G} \to M, \omega)$ be any regular, normal $G/P$ geometry. Then* $\dim(\mathfrak{inf}(\mathcal{G}, \omega)) \leq \dim(\mathfrak{a}^{\kappa_H(u)})$, *$\forall u$ in some open dense subset of $\mathcal{G}$.*

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Kruglikov-T. (2014): This holds $\forall u \in \mathcal{G}$.

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Fix $(G, P)$. Define $\mathfrak{U} := \max\{\dim(\mathfrak{a}^\phi) \mid 0 \neq \phi \in H^2_+(\mathfrak{g}_-, \mathfrak{g})\}$.

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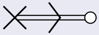
*Exception list (G simple):*  ,  ,  .

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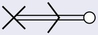
*Exception list ($G$ simple):*  ,  ,  .

**Theorem (Computability)**

*If $G/P$ is complex or split-real, can read $\mathfrak{U}$ from a Dynkin diagram.*