Symmetry gaps for geometric structures (Lecture 2)

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Lecture 2 Outline

1. Cartan geometry
2. Tanaka prolongation

Towards Cartan geometry

Klein geometry $(G \to G/P, \omega_G)$

$\overset{\text{(curvature)}}{\rightsquigarrow}$

Cartan geometry $(\mathcal{G} \to M, \omega)$

$\uparrow$ (generalize)

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Euclidean geometry $(\mathbb{R}^n, g_0)$

$\overset{\text{(curvature)}}{\rightsquigarrow}$

Riemannian geometry (M^n, g)

Cartan geometries

Let G be a Lie group, $P \leq G$ a closed subgroup.

Definition

A Cartan geometry $(\mathcal{G} \to M, \omega)$ of type (G, P) consists of a (right) principal P-bundle $\mathcal{G} \to M$ with a Cartan connection $\omega \in \Omega^1(\mathcal{G}; \mathfrak{g})$:

1. *ω is a coframing: $\omega_u : T_u\mathcal{G} \to \mathfrak{g}$ linear iso $\forall u \in \mathcal{G}$;*
2. *ω is P-equivariant: $R_p^*\omega = \mathrm{Ad}_{p^{-1}} \circ \omega$, $\forall p \in P$;*
3. *ω reproduces fund. vertical v.f.: $\omega(\zeta_A) = A$, $\forall A \in \mathfrak{p}$, where*

$$\zeta_A(u) = \frac{d}{dt}\bigg|_{t=0} u \cdot \exp(tA).$$

Example

Flat model: $(G \to G/P, \omega_G)$, where ω_G is the Maurer–Cartan form on G, i.e. $\omega_G(g) = (L_{g^{-1}})_*$. MC eqn: $d\omega_G + \frac{1}{2}[\omega_G, \omega_G] = 0$.

Symmetry algebra: $\mathfrak{inf}(\mathcal{G}, \omega) = \{\xi \in \mathfrak{X}(\mathcal{G})^P : \mathcal{L}_\xi \omega = 0\}$.

Curvature

Curvature: $K = d\omega + \frac{1}{2}[\omega, \omega] \in \Omega^2(\mathcal{G}; \mathfrak{g})$, i.e.

$$K(X, Y) = d\omega(X, Y) + [\omega(X), \omega(Y)].$$

- $K = 0$ ("flat") $\leftrightarrow$ locally equiv. to $(G \to G/P, \omega_G)$.
- K is P-equivariant: $R_p^* K = \mathrm{Ad}_{p^{-1}} \circ K$, $\forall p \in P$.
- K is horizontal, i.e. $K(\zeta_A, \cdot) = 0$, $\forall A \in \mathfrak{p}$. (Axiom 2 for ω implies $-\mathrm{ad}_A \circ \omega = \mathcal{L}_{\zeta_A}\omega = \iota_{\zeta_A} d\omega$.)

Definition (Curvature function)

$\kappa : \mathcal{G} \to \bigwedge^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$ *via* $\kappa(x, y) = K(\omega^{-1}(x), \omega^{-1}(y))$.

κ is P-equivariant, and codomain is a P-module. Ideally, impose P-inv. normalization conditions on κ to pin down ω uniquely.

Example

$(\mathcal{G} \to M, \omega)$ is torsion-free if κ is valued in $\bigwedge^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{p}$.

A fundamental theorem

Let $(G, P) = (\mathbb{E}(n), O(n))$ with Lie algebras $(\mathfrak{g}, \mathfrak{p}) = (\mathfrak{e}(n), \mathfrak{so}(n))$.
We have $\mathfrak{g} = \left\{ \begin{pmatrix} A & b \\ 0 & 0 \end{pmatrix} : A \in \mathfrak{so}(n), b \in \mathbb{R}^n \right\}$.

Theorem

There is an equivalence of categories btw Riemannian metrics and torsion-free Cartan geometries of type $(\mathbb{E}(n), O(n))$.

Write $\omega = \gamma + \theta \in \Omega^1(\mathcal{G}, \mathfrak{so}(n) \oplus \mathbb{R}^n)$. Then γ is the Levi-Civita connection and θ is called the soldering form.

Example ($n = 2$ case)

$$\omega = \begin{pmatrix} 0 & -\gamma & \theta^1 \\ \gamma & 0 & \theta^2 \\ 0 & 0 & 0 \end{pmatrix}, \; K = \begin{pmatrix} 0 & -d\gamma & d\theta^1 - \gamma \wedge \theta^2 \\ d\gamma & 0 & d\theta^2 + \gamma \wedge \theta^1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -c\theta^1 \wedge \theta^2 & 0 \\ c\theta^1 \wedge \theta^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

by torsion-freeness and horizontality. Thus, $\begin{cases} d\theta^1 = \gamma \wedge \theta^2 \\ d\theta^2 = -\gamma \wedge \theta^1 \\ d\gamma = c\, \theta^1 \wedge \theta^2 \end{cases}$.

Beyond metrics

Can consider other structures, e.g. conformal – i.e. $CO(n)$-str. But on the conformal frame bundle, normalization of str eqns does not pin down the coframe uniquely.

Strategy: build a new bundle. In general, one can imagine a tower of bundles...

This process is known as Cartan's equivalence method.

Motivated by his study of PDE, Cartan in 1910 gave a tour-de-force application of his equivalence method to study $(2,3,5)$-distributions. Nowadays, this "5-variables paper" is often referred to as the origin of Cartan connections. Associated to such G_2/P_1-geometries, a fundamental invariant emerged – a binary quartic field. (Modern perspective later...)

Tanaka theory & parabolic geometry

Cartan's equivalence method involves relatively easy steps – linear algebra and exterior differentiation. But carrying it out in practice, it is often "difficult to see the forest through the trees".

From the 1960's, the Japanese school (Tanaka and later Morimoto, Yamaguchi,...) developed further refinements of the method. In particular,

- Study of filtered manifolds / filtered G_0-structures.
- Tanaka prolongation – this important algebraic tool gives an upper bound for the sym dim.
- Harmonic theory & Fundamental curvature.

The study of parabolic geometries grew out of these ideas. The precise curvature normalization conditions to get equivalence to underlying geometric structures will be described in Lecture 2.

> For geometric structures that admit a Cartan geometric description, study the symmetry gap problem "upstairs".

Distributions & symbol algebras

Let $D^{-1} := D \subset TM$ (constant rank), form the weak-derived flag, i.e. $D^i = [D, D^{i+1}]$ for $i < 0$. Suppose $D^{-\nu} = TM$, $\exists \nu > 0$, i.e. "bracket-generating". Get a filtration

$$D =: D^{-1} \subset D^{-2} \subset ... \subset D^{-\nu} = TM.$$

Fix $x \in M$, take associated-graded: Let $\mathfrak{g}_i(x) := D^i(x)/D^{i+1}(x)$,

$$\mathfrak{m}(x) := \mathfrak{g}_{-1}(x) \oplus \mathfrak{g}_{-2}(x) \oplus ... \oplus \mathfrak{g}_{-\nu}(x).$$

The Lie bracket of v.f. induces a tensorial bracket on each $\mathfrak{m}(x)$, turning it into a nilpotent graded Lie algebra (NGLA) called the symbol algebra. We'll assume $\mathfrak{m}(x) \cong \mathfrak{m}$, $\forall x \in M$ as NGLA.

Filtered G_0-structures

Given (M, D) as before, one has a natural (graded) frame bundle:

$$F_{gr}(M) = \bigcup_{x \in M} \{u : \mathfrak{m} \to \mathfrak{m}(x) \text{ NGLA iso.}\}$$

This has structure group $\mathrm{Aut}_{gr}(\mathfrak{m})$, with Lie algebra:
$\mathfrak{der}_{gr}(\mathfrak{m}) \hookrightarrow \mathfrak{gl}(\mathfrak{g}_{-1})$ since $\mathfrak{g}_{-1}$ generates $\mathfrak{m} = \mathfrak{g}_-$.

Can specify reduction: $G_0 \leq \mathrm{Aut}_{gr}(\mathfrak{m})$, so $\mathfrak{g}_0 \leq \mathfrak{der}_{gr}(\mathfrak{m})$.

Analogous to $O(n)$-structures (metrics) as reductions of the frame bundle $F(M)$, we can analogously define filtered G_0-structures as G_0-reductions $\mathcal{G}_0 \subset F_{gr}(M)$.

Examples

Example (Riemannian geometry)

$\mathfrak{m} = \mathfrak{g}_{-1} \cong \mathbb{R}^n$ (abelian) and $\mathfrak{g}_0 = O(n) \leq \mathrm{Aut}_{gr}(\mathfrak{m}) \cong \mathrm{GL}(n;\mathbb{R})$.

Example (2nd order ODE: $D = \langle \partial_x + p\partial_y + f\partial_p \rangle \oplus \langle \partial_p \rangle$)

$\mathfrak{m} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2} = \langle e_1, e_2 \rangle \oplus \langle e_3 \rangle$ with $[e_1, e_2] = e_3$ (Heisenberg). Have splitting $\mathfrak{g}_{-1} = L_1 \oplus L_2$ and $\mathfrak{g}_0 =$ rescalings along L_1 and L_2 (2-dim). (Here, $\mathfrak{der}_{gr}(\mathfrak{m}) \cong \mathfrak{csp}(2,\mathbb{R}) \cong \mathfrak{gl}(2,\mathbb{R})$.)

Example ((2, 3, 5)-dist. $D = \langle D_x := \partial_x + p\partial_y + q\partial_p + f\partial_z, \partial_q \rangle$)

$$T := [\partial_q, D_x] = \partial_p + f_q\partial_z \neq 0, \quad [\partial_q, T] = f_{qq}\partial_z,$$
$$[T, D_x] = \partial_y + S\partial_z, \quad S = f_p + f_q f_z - D_x(f_q).$$

Thus, $\mathfrak{m} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-3} = \langle e_1, e_2 \rangle \oplus \langle e_3 \rangle \oplus \langle e_4, e_5 \rangle$, where:

$$[e_1, e_2] = e_3, \quad [e_1, e_3] = e_4, \quad [e_2, e_3] = e_5.$$

Tanaka prolongation

Tanaka prolongation: Given NGLA $\mathfrak{m}$ and $\mathfrak{g}_0 \leq \mathfrak{der}_{gr}(\mathfrak{m})$, define $pr(\mathfrak{m}, \mathfrak{g}_0)$ as the GLA s.t.

1. $pr_{\leq 0}(\mathfrak{m}, \mathfrak{g}_0) = \mathfrak{m} \oplus \mathfrak{g}_0$.
2. If $X \in pr_+(\mathfrak{m}, \mathfrak{g}_0)$ s.t. $[X, \mathfrak{g}_{-1}] = 0$, then $X = 0$.
3. $pr(\mathfrak{m}, \mathfrak{g}_0)$ is maximal among all GLA satisfying (1) and (2).

Special case: When $\mathfrak{g}_0 = \mathfrak{der}_{gr}(\mathfrak{m})$, we just write $pr(\mathfrak{m})$.

Theorem

$pr(\mathfrak{m}, \mathfrak{g}_0)$ is unique up to isomorphism.

Theorem

$\dim(pr(\mathfrak{m}, \mathfrak{g}_0))$ is an upper bound for the symmetry algebra of a filtered G_0-structure.

IDEA: Positive parts of this algebraic prolongation correspond to the geometric tower of bundles: $\ldots \to \mathcal{G}_2 \to \mathcal{G}_1 \to \mathcal{G}_0 \to M$.

Computing the Tanaka prolongation

Computing $pr(\mathfrak{m}, \mathfrak{g}_0)$ is done by an iterative process. For $r > 0$, suppose $\mathfrak{m}_r := \mathfrak{m} \oplus \mathfrak{g}_0 \oplus ... \oplus \mathfrak{g}_r$ is known. Then

$$\mathfrak{g}_{r+1} = \{A \in \mathfrak{gl}_{r+1}(\mathfrak{m}_r) : A[x,y] = [Ax,y] + [x,Ay], \forall x,y \in \mathfrak{m}\},$$

where $\mathfrak{gl}_{r+1}(\mathfrak{m}_r)$ consists of degree $r+1$ maps.
(One must further specify brackets to satisfy Jacobi.)

Example (Surface metrics)

$\mathfrak{m} = \mathfrak{g}_{-1} = \langle e_1, e_2 \rangle$, $\mathfrak{g}_0 = \mathfrak{so}(2) \ni T = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Let $A : \mathfrak{g}_{-1} \to \mathfrak{g}_0$.
Write $Ae_1 = aT$ and $Ae_2 = bT$. Require:

$$A[e_1, e_2] = [Ae_1, e_2] + [e_1, Ae_2].$$

Since $\mathfrak{g}_{-1}$ is abelian, then $0 = aT(e_2) - bT(e_1) = -ae_1 - be_2$, so $A = 0$. Thus, $pr(\mathfrak{m}, \mathfrak{g}_0) = \mathfrak{m} \oplus \mathfrak{g}_0$.

Exercise

Compute $pr(\mathfrak{m}, \mathfrak{g}_0)$ *for 2nd order ODE by-hand. Answer:* $\mathfrak{sl}_3$.

Show Maple file.

Tanaka prolongation revisited

Computing Tanaka prolongation directly as above is involved. However, if one already has a candidate $\mathfrak{g}$, it becomes simpler:

Q1: Given GLA $\mathfrak{g}$ with non-pos. part $(\mathfrak{m}, \mathfrak{g}_0)$, is $\mathfrak{g} \cong pr(\mathfrak{m}, \mathfrak{g}_0)$?
Q2: Given GLA $\mathfrak{g}$ with negative part $\mathfrak{m}$, is $\mathfrak{g} \cong pr(\mathfrak{m})$?

A1: True iff $H^1_+(\mathfrak{g}_-, \mathfrak{g}) = 0$.
A2: True iff $H^1_{\geq 0}(\mathfrak{g}_-, \mathfrak{g}) = 0$.

Here, $H^1(\mathfrak{g}_-, \mathfrak{g})$ refers to a Lie algebra cohomology group. The subscript refers to the induced grading on these spaces. In the "parabolic setting", we can efficiently compute this via Kostant's theorem (see Lecture 4).