Symmetry gaps for geometric structures (Lecture 1)

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The symmetry gap problem

For a given type of geometric structure,
what is the gap between maximal and
submaximal (infinitesimal) symmetry dimensions $\mathfrak{S}$?

Example

For Riemannian metrics on surfaces: max $= 3$, but $\mathfrak{S} = 1$. Note: We focus on the *local* problem. (Cylinders are locally flat.)

Numerous classical studies: Lie, Tresse, Fubini, Cartan, Yano, Wakakuwa, Vranceanu, Egorov, Obata, Kobayashi, Nagano,...

Many recent studies: Čap, Neusser, Kruglikov, Winther, Matveev, Isaev, Zalabova, Doubrov, de Medeiros, ...

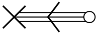
Q: How to get a priori upper bounds? Are these bounds sharp?

Syllabus

This lecture series will focus on the "symmetry gap problem".

Things you'll learn about along the way:

- Cartan geometries, e.g. Riemannian and parabolic geometries
- Tanaka prolongation
- Kostant's theorem for computing Lie algebra cohomology
- In favourable circumstances, encoding symmetry gaps on Dynkin diagrams, e.g.



$$\Rightarrow \quad \mathfrak{S} = 7.$$

(Note: A priori, no assumptions on local homogeneity of submaximally symmetric models, despite algebraic character of the final result.)

Reference:

- B. Kruglikov & D. The, "The gap phenomenon in parabolic geometries", J. Reine Angew. Math. (2014).

Lecture 1 outline

1. Examples
2. The symmetry gap for surface metrics

Example: Riemannian geometry

Example (Riemannian manifolds (M^n, g))

- A symmetry is a diffeomorphism $\phi : M \to M$ s.t. $\phi^* g = g$.
- Infinitesimally, a symmetry ("Killing v.f.") $X \in \mathfrak{X}(M)$ satisfies $\mathcal{L}_X g = 0$. Locally, $g = g_{ab} dx^a \otimes dx^b$, this is a linear PDE in $X = X^a \partial_{x^a}$: $X^c \partial_{x^c} g_{ab} + (\partial_{x^a} X^c) g_{cb} + (\partial_{x^b} X^c) g_{ac} = 0$.
- sym. dim. $\leq \binom{n+1}{2}$. Sharp on constant curvature spaces:
  - $\mathbb{R}^n \cong \mathbb{E}(n)/O(n)$.
  - $S^n \cong O(n+1)/O(n)$.
  - $H^n \cong O(n,1)/O(n)$.
- Let $\mathfrak{S}$ = next possible sym dim among models not locally isometric to a constant curvature space:

n	max	submax	Citation
2	3	1	Darboux / Koenigs (∼1890)
3	6	4	Wang (1947)
4	10	8	Egorov (1955)
≥ 5	$\binom{n+1}{2}$	$\binom{n}{2}+1$	Wang (1947), Egorov (1949)

Examples: Parabolic geometries

Let G is a semisimple Lie group and P is a parabolic subgroup. (Examples later.)

- IDEA: Parabolic geometries are "curved versions" of G/P. (More details later.)
- Maximally symmetric model is locally unique, has sym alg $\mathfrak{g}$.

Examples

	Max	$\mathfrak{g}$	$\mathfrak{S}$
Lorentzian conf. str. $(M^4, [g])$	15	$\mathfrak{so}(2,4)$	7
$(2,3,5)$-distributions	14	G_2	7
scalar 2nd order ODE	8	$\mathfrak{sl}_3$	3

Other structures: projective, conformal, CR, 2nd order systems, Legendrian contact, Segré, G-contact, various classes of generic distributions,....

Not parabolic: Riemannian, affine, symplectic, contact, ...

Example: 2nd order ODE

$y'' = f(x, y, y')$. Point transformations: $\begin{cases} \tilde{x} = \tilde{x}(x, y) \\ \tilde{y} = \tilde{y}(x, y) \end{cases}$. Letting $p = y'$ and $q = y''$, we can prolong:

$$\tilde{p} = \frac{\tilde{y}_x + p\tilde{y}_y}{\tilde{x}_x + p\tilde{x}_y}, \quad \tilde{q} = \frac{\tilde{p}_x + p\tilde{p}_y + q\tilde{p}_p}{\tilde{x}_x + p\tilde{x}_y}.$$

Symmetries are vector fields on (x, y)-space whose prolongation to (x, y, p, q)-space (second jet-space $J^2(\mathbb{R}, \mathbb{R})$) are tangent to the submanifold $q = f(x, y, p)$.

Example

$y'' = 0$ has max sym $\mathfrak{g} = \mathfrak{sl}_3$. Have $\mathfrak{S} = 3$, e.g. $y'' = \exp(y')$.

Example: 2nd order ODE continued

Reformulation: Consider (x, y, p, q)-space equipped with $\langle dy - pdx,\ dp - qdx \rangle$. On $q = f(x, y, p)$, get a line field:

$$E = \langle \partial_x + p\partial_y + f\partial_p \rangle.$$

Wrt point transformations, another line field is distinguished:

$$V = \langle \partial_p \rangle.$$

Geometric structure: Let M be (x, y, p)-space with a contact distribution $D = \ker\{dy - pdx\} = \langle \partial_x + p\partial_y, \partial_p \rangle$. A 2nd order ODE is encoded by the data of a splitting $D = E \oplus V$ as above.

Note $[D, D] = D + span\{\partial_y + f_p\partial_p\} = TM$.

A symmetry is $X \in \mathfrak{X}(M)$ s.t. $\mathcal{L}_X E \subset E$ and $\mathcal{L}_X V \subset V$.

Example: 2nd order ODE continued

Example

$y'' = exp(y')$ has point syms

$$\partial_x, \quad \partial_y, \quad x\partial_x + (y-x)\partial_y - \partial_p.$$

These are projectable over (x,y)-space. Usually write only $x\partial_x + (y-x)\partial_y$ for the last one.

Prolongation:

- Given X on (x,y)-space, can prolong to $X^{(1)}$ on (x,y,p)-space by requiring $\mathcal{L}_{X^{(1)}}(dy - pdx) \in \langle dy - pdx \rangle$.
- Can further prolong to $X^{(2)}$ on (x,y,p,q)-space, by further requiring $\mathcal{L}_{X^{(2)}}(dp - qdx) \in \langle dy - pdx, dp - qdx \rangle$.

Exercise

The v.f. above prolong to $\partial_x,\ \partial_y,\ x\partial_x + (y-x)\partial_y - \partial_p - q\partial_q$ (that are tangent to $q = \exp(p)$).

$(2,3,5)$-distribution: $(M^5, D \subset TM)$ with rank growth $(2,3,5)$ for:

$$D \subset [D,D] \subset [D,[D,D]] = TM.$$

Monge form $z' = f(x,y,y',y'',z)$: In (x,y,p,q,z)-space, let

$$\begin{aligned} D &= \ker\{dy - pdx,\ dp - qdx,\ dz - fdx\} \\ &= span\{\partial_x + p\partial_y + q\partial_p + f\partial_z,\ \partial_q\}. \end{aligned}$$

where $f = f(x,y,p,q,z)$ satisfies $f_{qq} \neq 0$.

Example

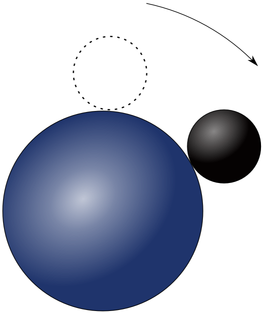
Have 14-dim sym for $f = q^2$ ("Hilbert–Cartan eqn": $z' = (y'')^2$); $\mathfrak{S} = 7$, e.g. $f = q^3$.

A symmetry $X \in \mathfrak{X}(M)$ satisfies $\mathcal{L}_X D \subset D$.

Rolling distributions and G_2

Consider a 2-sphere rolling on another without twisting or slipping.

- Configuration space M is 5-dimensional.
- No twisting or slipping $\Rightarrow$ constraints on velocity space TM. Get rank 2 distribution $D \subset TM$ of allowable directions.



Let $\rho \geq 1$ be the ratio of the radii.
If $\rho \neq 1$, get $(2, 3, 5)$-geometry.

- $\rho \neq 3$: $\mathfrak{so}(3) \times \mathfrak{so}(3)$ symmetry
- $\rho = 3$: (split) $\mathfrak{g}_2$ symmetry
  (Bryant, Zelenko, Bor–Montgomery, Baez–Huerta)

Show Maple file.

Approaches to the symmetry gap problem

Various classical approaches:

- Study integrability conditions for equation defining symmetry.
- Classification of Lie algebras of vector fields in the plane / space, find invariant structures.
- Cartan's method of equivalence & Cartan reduction method.

These methods are heavily case-dependent, are restricted to low dimensions, or often place additional restrictions ("locally constant type") on the problem.

We'll emphasize the Cartan approach, since the solution of the Cartan equivalence problem for wide classes of geometric structures has been worked out, e.g. Riemannian, affine, parabolic geometries, ODE, ...

> Goal: Motivate Cartan geometry formulation of the "solution of the equivalence problem" via surface metrics.

Local equivalence of Riemannian metrics

Q: Does $\exists \varphi : (M^n, g) \to (\tilde{M}^n, \tilde{g})$ s.t. $\varphi^* \tilde{g} = g$?

Locally diagonalize, e.g. $g = (\theta^1)^2 + \ldots + (\theta^n)^2$, get o.n. coframes $\{\theta^i\}$ and $\{\tilde{\theta}^i\}$. Reformulate Q as a Cartan equivalence problem:

Q: Does $\exists \varphi : M \to \tilde{M}$ s.t. $\varphi^* \tilde{\theta}^i = g^i_j(x)\theta^j$ for $g : M \to O(n)$?

IDEA: Build a bundle that incorporates the ambiguity, try to find a canonical coframing there.

In the literature, "soln of the Cartan equivalence problem" refers to finding such a canonical coframing on some bundle over M.

Metrics $\rightsquigarrow$ orthonormal frame bundle $F_{on}(M)$.

The orthonormal frame bundle

Let $\dim(M) = n$.

- A frame at $x \in M$ is a linear iso. $u : \mathbb{R}^n \to T_x M$. A coframe is the inverse map $u^{-1} : T_x M \to \mathbb{R}^n$.
- Frame bundle $\pi : F(M) \to M$. ($\pi^{-1}(x) =$ all frames at x.) This is a principal $\mathrm{GL}(n;\mathbb{R})$-bundle; right action $u \cdot a = u \circ a$.

Given (M^n, g), fix std metric $(\mathbb{R}^n, g_0)$, restrict $F(M)$ to $F_{on}(M)$, i.e. frames that are isometries. Choice of local o.n. coframing $\{\theta^i\}$:

$$g = (\theta^1)^2 + ... + (\theta^n)^2$$

is the same as a local section of $\pi : F_{on}(M) \to M$.

$n = 2$ case: $\dim(F_{on}(M)) = 3$. Local coords: (x, y) on M, fibre coord t (rotation angle). Outline:

- $\exists$ distinguished coframing ω on $F_{on}(M)$ (o.n. frame bundle)
- Discuss $rank(\omega)$ and its relation to symmetry.
- Establish that 2 is not possible.

Local coframing on $\mathcal{G} = F_{on}(M)$ for (M^2, g)

Write $g = (\bar{\theta}^1)^2 + (\bar{\theta}^2)^2$ wrt o.n. coframe $\bar{\theta}^1, \bar{\theta}^2 \in \Omega^1(M)$.
Lift to $\mathcal{G}$: Let $\begin{pmatrix} \theta^1 \\ \theta^2 \end{pmatrix} = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix} \begin{pmatrix} \bar{\theta}^1 \\ \bar{\theta}^2 \end{pmatrix}$.

Exercise

Show that $\begin{cases} d\theta^1 = -dt \wedge \theta^2 + A\theta^1 \wedge \theta^2 = -\gamma \wedge \theta^2 \\ d\theta^2 = +dt \wedge \theta^1 + B\theta^1 \wedge \theta^2 = +\gamma \wedge \theta^1 \end{cases}$,
where $\gamma = dt - A\theta^1 - B\theta^2$.

Apply d: $\begin{cases} 0 = d^2\theta^1 = -d\gamma \wedge \theta^2 \\ 0 = d^2\theta^2 = +d\gamma \wedge \theta^1 \end{cases}$, so $d\gamma = c\,\theta^1 \wedge \theta^2$. Get

structure eqns $\begin{cases} d\theta^1 = \gamma \wedge \theta^2 \\ d\theta^2 = -\gamma \wedge \theta^1 \\ d\gamma = c\,\theta^1 \wedge \theta^2 \end{cases}$ for $\omega = (\theta^1, \theta^2, \gamma)$. Note that ω is pinned down uniquely by these structure eqns.

Coframe rank and symmetry

We saw $g \rightsquigarrow \exists! \omega$. Symmetries of g correspond to symmetries Φ of the coframing ω, i.e. $\Phi^* \omega = \omega$. Since $d \circ \Phi^* = \Phi^* \circ d$, then Φ preserves structure functions, e.g. $\Phi^* c = c$. Rinse & repeat:

$$dc = c_1 \theta^1 + c_2 \theta^2 + c_3 \gamma.$$

Then Φ preserves c_1, c_2, c_3. Keep going... The rank r of ω is the number of indep. fcns obtained via this process. General thm:

Theorem (c.f. Olver, "Equivalence, Invariants, Symmetry", Thm 8.22)

A coframe ω of rank $r \geq 0$ on an m-mfld has $\dim(sym) = m - r$.

Example

If c is constant, then $r = 0$ and we get str. eqns for a Lie alg/grp:

$$[e_i, e_j] = C_{ij}{}^{\ell} e_{\ell} \quad \Longleftrightarrow \quad d\omega^{\ell} = -\frac{1}{2} C_{ij}{}^{\ell} \omega^i \wedge \omega^j,$$

where $\{\omega^i\}$ is the dual basis to $\{e_i\}$.

Symmetry gap for surface metrics

Thm: Any (M^2, g) cannot have precisely 2 Killing vectors.

Proof: On $\mathcal{G} = F_{on}(M)$, we saw $\exists!$ coframing $\omega = (\theta^1, \theta^2, \gamma)$ with:

$$\begin{cases} d\theta^1 = \gamma \wedge \theta^2 \\ d\theta^2 = -\gamma \wedge \theta^1 \\ d\gamma = c\,\theta^1 \wedge \theta^2 \end{cases} \quad \Rightarrow \quad \begin{cases} 0 = d^2\theta^1 = d^2\theta^2 \\ 0 = d^2\gamma = dc \wedge \theta^1 \wedge \theta^2 \\ dc = f\theta^1 + g\theta^2 \end{cases}$$

Assuming $\dim(sym) = 2$, then $rank(\omega) = 1$, so c is nonconstant, and f, g fcns of c. Then

$$\begin{cases} 0 = d^2 c \wedge \theta^1 = f\gamma \wedge \theta^2 \wedge \theta^1 \\ 0 = d^2 c \wedge \theta^2 = g\gamma \wedge \theta^2 \wedge \theta^1 \end{cases} \quad \Rightarrow \quad f = g = 0 \Rightarrow c \text{ constant} \quad ⨳$$

This was easy, but working directly with str eqns in high dim is not. We'll reformulate this using the notion of Cartan geometry.