Quantum Harmonic Analysis and its Applications

jointly with Eirik Skrettingland
Nordfjordeid Summer School 2019
Analysis, Geometry and PDE

Franz Luef

Department of mathematical sciences, NTNU

July 1-5, 2019

A **quadratic** time-frequency representation is a sesquilinear form $Q(\phi, \psi)$ satisfying

$$\begin{aligned} Q(a\phi + b\psi, a\phi + b\psi) &= |a|^2 Q(\phi, \phi) + |b|^2 Q(\psi, \psi) \\ &+ a\overline{b}\, Q(\phi, \psi) + \overline{a}b\, Q(\psi, \phi). \end{aligned}$$

Cohen class

A quadratic time-frequency distribution Q is said to be of *Cohen's class* if Q is given by

$$Q(\psi) = Q_\phi(\psi) := W(\psi, \psi) * \phi$$

for some $\phi \in \mathcal{S}'(\mathbb{R}^{2d})$.

The class of functions ψ to which we may apply Q_ϕ clearly depends on the distribution ϕ. The Wigner distribution is obtained by picking $\phi = \delta_0$, where δ_0 is Dirac's delta distribution centered at 0.

Lemma

Let Q be a quadratic time-frequency distribution satisfying

1. $Q(\pi(z)\psi) = T_z(Q(\psi))$,
2. $|Q(\psi_1, \psi_2)(0)| \leq \|\psi_1\|_2 \|\psi_2\|_2$,

for all $z \in \mathbb{R}^{2d}$ and $\psi_1, \psi_2 \in L^2(\mathbb{R}^d)$.
Then $Q(\psi) = W(\psi, \psi) * \phi$ for some $\phi \in \mathcal{S}'(\mathbb{R}^{2d})$.

By using the connection between Cohen's class and convolutions of operators we obtain a weak uncertainty principle for Cohen's class distributions. The result is modeled on uncertainty principles for the spectrogram and Wigner distribution.

Proposition

Let $S \in B(L^2(\mathbb{R}^d))$ and let Q_S be the Cohen's class distribution determined by $Q_S(\psi) = (\psi \otimes \psi) \star S$ for $\psi \in L^2(\mathbb{R}^d)$. If $\Omega \subset \mathbb{R}^{2d}$ is a measurable subset such that

$$\iint_\Omega |Q_S(\psi)| \, dz \geq (1-\epsilon)\|S\|_{B(L^2(\mathbb{R}^d))}$$

for some $\psi \in L^2(\mathbb{R}^d)$ with $\|\psi\|_{L^2} = 1$ and $\epsilon \geq 0$, then

$$\mu(\Omega) \geq 1 - \epsilon.$$

We give a characterization of Cohen's class as convolutions with a fixed operator.

Cohen class characterization

For $\phi \in \mathcal{S}'(\mathbb{R}^{2d})$, the associated Cohen's class distribution Q_ϕ is given by

$$Q_\phi(\psi) = (\psi \otimes \psi) \star L_\phi \text{ for } \psi \in \mathcal{S}(\mathbb{R}^d), \tag{1}$$

where L_ϕ is the Weyl transform of ϕ.
Conversely, any operator $A \in \mathfrak{S}'$ determines a Cohen's class distribution by

$$Q_A(\psi) := (\psi \otimes \psi) \star \check{A} \text{ for } \psi \in \mathcal{S}(\mathbb{R}^d).$$

Comments

1. This proposition shows that any shift-invariant, $Q(\pi(z)\psi) = T_z(Q(\psi))$ for $z \in \mathbb{R}^{2d}$ and $\psi \in L^2(\mathbb{R}^d)$, weakly continuous quadratic time-frequency distribution is given by a convolution with a fixed operator on $L^2(\mathbb{R}^d)$.
2. Hence, any Cohen's class distribution may be described by either a distribution $\phi \in \mathcal{S}'(\mathbb{R}^{2d})$ or by an operator $A \in \mathfrak{S}'$, where
$$Q_\phi = Q_A \text{ if } L_\phi = \check{A}.$$
3. We have defined Q_A in terms of $\check{A}$ to simplify formulas, note that A and $\check{A}$ share all relevant properties, such as positivity, trace and membership of Schatten classes.
4. Using that $Q_\phi(\psi) = (\psi \otimes \psi) \star L_\phi$, we may apply the theory of convolutions of operators to deduce some simple results on Cohen's class distributions.

Examples

1. The Wigner distribution $Q_\phi(\psi) = W(\psi, \psi)$ is given by $\phi = \delta_0$. $W(\psi, \psi)$ is also given by

$$W(\psi, \psi) = (\psi \otimes \psi) \star L_{\delta_0}$$

for $\psi \in \mathcal{S}(\mathbb{R}^d)$. By a result of Grossmann, $L_{\delta_0} = 2^d P$, where P is the parity operator.

2. Fix a window $\varphi \in L^2(\mathbb{R}^d)$ and consider the operator $S = \varphi \otimes \varphi$. Then $\check{S} = \check{\varphi} \otimes \check{\varphi}$ defines a Cohen's class distribution Q_S by

$$Q_S(\psi) = (\psi \otimes \psi) \star (\check{\varphi} \otimes \check{\varphi}) = |V_\varphi \psi|^2.$$

This Cohen's class distribution is therefore the spectrogram. The corresponding function ϕ, i.e. the Weyl symbol of $\check{\varphi} \otimes \check{\varphi}$, is the Wigner distribution $W(\check{\varphi}, \check{\varphi})$.

Proposition

Fix $1 \leq p \leq \infty$. Consider a Cohen's class distribution Q_ϕ for $\phi \in \mathcal{S}'(\mathbb{R}^{2d})$. Let L_ϕ be the Weyl transform of ϕ.
If $L_\phi \in \mathcal{T}^p$, then $Q_\phi(\psi)$ is well-defined for any $\psi \in L^2(\mathbb{R}^d)$ and $Q_\phi(\psi) \in L^p(\mathbb{R}^{2d})$ with $\|Q(\psi)\|_{L^p} \leq \|\psi\|_{L^2}^2 \|S\|_{\mathcal{T}^p}$.
In particular, if $L_\phi \in B(L^2(\mathbb{R}^d))$, then $Q_\phi(\psi) \in L^\infty(\mathbb{R}^{2d})$ with $\|Q(\psi)\|_{L^\infty} \leq \|\psi\|_{L^2}^2 \|S\|_{B(L^2(\mathbb{R}^d))}$.

Remark

By Pool's Theorem, the condition that $L_\phi \in \mathcal{T}^2$ is equivalent to $\phi \in L^2(\mathbb{R}^{2d})$. Unfortunately there is no equally simple characterization of those ϕ such that $L_\phi \in \mathcal{T}^1$ or $L_\phi \in B(L^2(\mathbb{R}^d))$.

Question

Given a Cohen's class distribution Q_ϕ, one might ask whether any $\psi \in L^2(\mathbb{R}^d)$ is uniquely determined by $Q_\phi(\psi)$.

Since ψ enters the expression for $Q_\phi(\psi)$ via $\psi \otimes \psi$, we can at most hope that $\psi \otimes \psi$ is uniquely determined by $Q_\phi(\psi)$. It is simple to show that $\psi_1 \otimes \psi_1 = \psi_2 \otimes \psi_2$ if and only if $\psi_1 = e^{i\alpha}\psi_2$ for some $\alpha \in \mathbb{R}$, so we will ask whether ψ is determined by $Q_\phi(\psi)$ up to some constant phase $e^{i\alpha}$ with $\alpha \in \mathbb{R}$.

In the special case where $L_\phi \in \mathcal{T}^1$, a rather weak condition on ϕ is enough to ensure this.

Phase retrieval

Let $\phi \in L^2(\mathbb{R}^{2d})$ be a function such that the Weyl transform L_ϕ is trace class. Assume that the set $\{z \in \mathbb{R}^{2d} : \mathcal{F}_\sigma \phi(z) = 0\}$ has dense complement in $\mathbb{R}^{2d}$. If $Q_\phi(\psi_1) = Q_\phi(\psi_2)$ for $\psi_1, \psi_2 \in L^2(\mathbb{R}^d)$, then $\psi_1 = e^{i\alpha}\psi_2$ for some constant $\alpha \in \mathbb{R}$.

We say that a Cohen's class distribution Q_ϕ is **positive** if $Q_\phi(\psi)(z) \geq 0$ for all $z \in \mathbb{R}^{2d}$ and ψ in the domain of Q_ϕ.

Positivity

Let Q_ϕ be a Cohen's class distribution such that the Weyl transform L_ϕ is bounded on $L^2(\mathbb{R}^d)$. Then Q_ϕ is positive if and only if L_ϕ is a positive operator.

We may equivalently ask which conditions ϕ must satisfy to ensure that the Weyl transform L_ϕ is a positive operator.
This question is of interest in quantum mechanics, and providing a general answer has turned out to be difficult. The so-called KLM conditions due to Kastler, Loupias and Miracle-Sole.

We say that a Cohen's class distribution Q_ϕ has the **correct total energy property** if

$$\iint_{\mathbb{R}^{2d}} Q_\phi(\psi)(z)\, dz = \|\psi\|_{L^2}^2$$

for all $\psi \in L^2(\mathbb{R}^d)$. One might think of $Q_\phi(\psi)$ as an energy distribution for the signal ψ, and so one would hope that the total energy $\|\psi\|_{L^2}^2$ equals the integral of the energy distribution $Q_\phi(\psi)$.

Proposition

Let Q_ϕ be a Cohen's class distribution, and let L_ϕ be the Weyl transform of ϕ. If $L_\phi \in \mathcal{T}^1$, then

$$\iint_{\mathbb{R}^{2d}} Q_\phi(\psi)\, dz = \|\psi\|_{L^2}^2 \operatorname{tr}(L_\phi) \qquad \text{for any } \psi \in L^2(\mathbb{R}^d). \tag{2}$$

If in addition $\phi \in L^1(\mathbb{R}^{2d})$, then

$$\iint_{\mathbb{R}^{2d}} Q_\phi(\psi)\, dz = \|\psi\|_{L^2}^2 \iint_{\mathbb{R}^{2d}} \phi(z)\, dz \tag{3}$$

Proposition

Let Q_ϕ be a Cohen's class distribution. Q_ϕ is positive and has the correct total energy property if and only if the Weyl transform L_ϕ is a positive trace class operator with $\operatorname{tr}(L_\phi) = 1$. If this is the case, there exists an orthonormal basis $\{\varphi_n\}_{n\in\mathbb{N}}$ in $L^2(\mathbb{R}^d)$ and a sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ of positive numbers with $\sum_{n=1}^\infty \lambda_n = 1$ such that

$$Q_\phi(\psi)(z) = \sum_{n=1}^{\infty} \lambda_n |V_{\varphi_n}\psi|^2(z),$$

and this sum converges uniformly for each $\psi \in L^2(\mathbb{R}^d)$.

A restatement of the previous theorem is that the Cohen's class distributions Q that are positive with the correct total energy property are exactly given by

$$Q(\psi) = (\psi \otimes \psi) \star S_Q$$

for some positive operator $S_Q \in \mathcal{T}^1$ with $\operatorname{tr}(S_Q) = 1$.

Examples

1. The spectrogram $|V_\varphi \psi(z)|^2 = (\psi \otimes \psi) \star (\check{\varphi} \otimes \check{\varphi})$ for $\varphi \in L^2(\mathbb{R}^d)$ with $\|\varphi\|_{L^2} = 1$ is both positive and has the correct total energy property. This agrees with a prior result since the operator $\check{\varphi} \otimes \check{\varphi}$ is positive and $\mathrm{tr}(\check{\varphi} \otimes \check{\varphi}) = \langle \check{\varphi}, \check{\varphi} \rangle = 1$.
2. The Wigner distribution $W(\psi) = (\psi \otimes \psi) \star 2^d P$ is not positive, as P is not a positive operator. The correct total energy property holds for some, but not all $\psi \in L^2(\mathbb{R}^d)$.
3. Using a result due to Gracia-Bondía and Várilly, we may now give a characterization of the Gaussians that give positive Cohen's class distributions with the correct total energy property. To make this precise, let Φ_M be the normalized Gaussian

$$\Phi_M(z) = 2^n \frac{1}{\det(M)^{1/4}} e^{-z^T \cdot M \cdot z} \text{ for } z \in \mathbb{R}^{2d},$$

where M is a $2d \times 2d$-matrix.

The result of Gracia-Bondia and Varilly states that the Weyl transform L_{Φ_M} is a positive trace class operator if and only if

$$M = S^T \Lambda S,$$

where S is a symplectic matrix and Λ is diagonal matrix of the form

$$\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots \lambda_d, \lambda_1, \lambda_2, \ldots, \lambda_d)$$

with $0 < \lambda_i \leq 1$. Hence these Gaussians Φ_M are exactly the Gaussians such that the Cohen's class distribution Q_{Φ_M} is positive with the correct total energy property.

Note that this provides examples of positive Cohen's class distributions with the correct total energy property that are *not* spectrograms, since some of the Gaussians above do not correspond to operators of the form $\check{\varphi} \otimes \check{\varphi}$ under the Weyl transform.

These results are also linked with the symplectic structure of the phase space.

We saw that S defines a Cohen's class distribution Q_S by $Q_S(\psi)(z) = ((\psi \otimes \psi) \star \check{S})(z)$. In fact, there is a close connection between operators $f \star S$ and Cohen class distribution $Q_{\check{S}}$.

Proposition

Let $S \in \mathcal{T}^1$, $f \in L^\infty(\mathbb{R}^{2d})$ and $\psi \in L^2(\mathbb{R}^d)$. Let $Q_{\check{S}}$ be the Cohen's class distribution $Q_S(\psi) = (\psi \otimes \psi) \star \check{S}$. Then

$$\langle f \star S, \psi \otimes \psi \rangle_{B(L^2),\mathcal{T}^1} = \langle f, Q_S(\psi) \rangle_{L^\infty, L^1}.$$

More explicitly

$$\langle (f \star S)\psi, \psi \rangle = \iint_{\mathbb{R}^{2d}} f(z) Q_S(\psi)(z)\, dz. \quad (4)$$

1. If we pick $S = \varphi \otimes \varphi$ for $\varphi \in L^2(\mathbb{R}^d)$, then $S \in \mathcal{T}^1$ and $\check{S} = \check{\varphi} \otimes \check{\varphi}$. For $f \in L^\infty(\mathbb{R}^{2d})$ the operator $f \star S$ is the localization operator $\mathcal{A}_f^\varphi$ and the Cohen's class distribution determined by S is the spectrogram

$$Q_S(\psi)(z) = (\psi \otimes \psi) \star (\check{\varphi} \otimes \check{\varphi})(z) = |V_\varphi \psi(z)|^2.$$

$$\langle \mathcal{A}_f^\varphi \psi, \psi \rangle = \iint_{\mathbb{R}^{2d}} f(z) |V_\varphi \psi(z)|^2 \, dz.$$

2. For $S = 2^d P$ the Proposition describes the Weyl calculus and the Cohen's class distribution associated to $2^d P = \left(2^d P\right)\check{}$ is the Wigner distribution

$$Q_{2^d P}(\psi) = (\psi \otimes \psi) \star 2^d P(z) = W(\psi)(z).$$

For a function $f \in L^1(\mathbb{R}^{2d})$ the operator $f \star 2^d P$ is the Weyl transform L_f: the Weyl symbol of $2^d P$ is δ_0 and we have

$$\langle L_f \psi, \psi \rangle = \iint_{\mathbb{R}^{2d}} f(z) W(\psi)(z) \, dz$$

$$\langle \chi_\Omega \star S\psi, \psi \rangle = \iint_\Omega Q_S(\psi)(z)\, dz.$$

The right hand side of this equation may be interpreted as a measure of the concentration of the energy of ψ in the region Ω of the time-frequency plane, and leads to a

Localization problem for Cohen's class

For a Cohen's class distribution Q and a measurable $\Omega \subset \mathbb{R}^{2d}$. Find the signal $\psi \in L^2(\mathbb{R}^d)$ with $\|\psi\|_{L^2} = 1$ that maximizes

$$\iint_\Omega Q(\psi)(z)\, dz.$$

The first Equation implies that the problem is solved by considering the eigenfunctions of the operator $\chi_\Omega \star S$ by Courant's min-max principle.

Theorem

Let $\Omega \subset \mathbb{R}^{2d}$ be a measurable subset, let $S \in B(L^2(\mathbb{R}^d))$ be a selfadjoint operator and let Q_S be the associated Cohen's class distribution $Q_S(\psi) = (\psi \otimes \psi) \star \check{S}$. Assume that $\chi_\Omega \star S$ is a compact operator. Let $\lambda_1 \geq \lambda_2, ...$ be the positive eigenvalues of $\chi_\Omega \star S$ (counted with multiplicities) and let ϕ_i be the eigenvector corresponding to λ_i for $i \in \mathbb{N}$.
Then

$$\iint_\Omega Q_S(\phi_n)(z)dz = \max\left\{\iint_\Omega Q_S(\psi)(z)dz : \|\psi\|_2 = 1, \psi \perp \phi_1, ., \phi_{n-1}\right.$$

We have formulated the result by requiring that $\chi_\Omega \star S$ is compact. It is easy to find conditions making this true;it will be true if $\mu(\Omega) < \infty$ and $S \in \mathcal{T}^p$ for some $p < \infty$. However, $\chi_\Omega \star S$ may well be compact in other cases too.

Proof

By the min-max principle we know that

$$\lambda_n = \min_{\psi_1,\ldots,\psi_{n-1}} \max_{\substack{\psi \perp \psi_1,\ldots,\psi_{n-1} \\ \|\psi\|_{L^2}=1}} \langle (\chi_\Omega \star S)\psi, \psi \rangle,$$

where $\psi_1, \psi_2, ..., \psi_{n-1}$ is any set of linearly independent vectors in $L^2(\mathbb{R}^d)$.
Since $\lambda_n = \langle (\chi_\Omega \star S)\phi_n, \phi_n \rangle$ and $\phi_n \perp \phi_1, ...\phi_{n-1}$, the minimum is achieved when $\psi_1 = \phi_1, \psi_2 = \phi_2, ..., \psi_{n-1} = \phi_{n-1}$, hence

$$\lambda_n = \max_{\substack{\psi \perp \phi_1,\ldots,\phi_{n-1} \\ \|\psi\|_{L^2}=1}} \langle (\chi_\Omega \star S)\psi, \psi \rangle.$$

We know that $\langle (\chi_\Omega \star S)\psi, \psi \rangle = \iint_\Omega Q_S(\psi)(z)\, dz$, and since $\lambda_n = \langle (\chi_\Omega \star S)\phi_n, \phi_n \rangle$ we get

$$\iint_\Omega Q_S(\phi_n)(z)dz = \max \left\{ \iint_\Omega Q_S(\psi)(z)dz : \|\psi\|_{L^2} = 1, \psi \perp \phi_1, ., \phi_{n-1}$$

The interpretation of

$$\iint_{\Omega} Q_{\check{S}}(\psi)(z)\, dz$$

as the energy concentration of ψ in Ω is more natural when $Q_{\check{S}}$ is positive and normalized in the sense that

$$\iint_{\mathbb{R}^{2d}} Q(\psi)(z)\, dz = \|\psi\|_{L^2}^2.$$

This is satisfied exactly when $S \in \mathcal{T}^1$ is a positive operator with $\mathrm{tr}(S) = 1$.
In this case the operators $\chi_\Omega \star S$ are the mixed-state localization operators.

Accumulated spectrogram

We will denote the eigenvalues of $\chi_\Omega \star S$ by $\{\lambda_k^\Omega\}_{k\in\mathbb{N}}$ and the orthonormal basis formed by its eigenfunctions by $\{h_k^\Omega\}_{k\in\mathbb{N}}$, thus the spectral representation is

$$\chi_\Omega \star S = \sum_{k=1}^{\infty} \lambda_k^\Omega h_k^\Omega \otimes h_k^\Omega.$$

We always assume that the eigenvalues are in decreasing order, i.e. $\lambda_1^\Omega \geq \lambda_2^\Omega \geq \dots$.

Lemma

For a $S \in \mathcal{T}^1$ we denote by $\tilde{S} := S \star \check{S}$.
$\tilde{S}$ is a positive, continuous function such that

$$\int_{\mathbb{R}^{2d}} \tilde{S}(z)\, dz = \mathrm{tr}(S)\mathrm{tr}(S) = 1.$$

In the special case where $S = \varphi \otimes \varphi$ for some $\varphi \in L^2(\mathbb{R}^d)$, we get that $\tilde{S}(z) = |V_\varphi \varphi(z)|^2$.

Proposition

Let S be a density operator and let $\Omega \subset \mathbb{R}^{2d}$ be a compact set. Then

$$\chi_\Omega * \tilde{S}(z) = \sum_{k=1}^{\infty} \lambda_k^\Omega Q_S(h_k^\Omega)(z), \quad \text{for } z \in \mathbb{R}^{2d}.$$

Proof

$\chi_\Omega * \tilde{S} = \chi_\Omega * (S \star \check{S}) = (\chi_\Omega \star S) \star \check{S}$ in $L^1(\mathbb{R}^{2d}) \cap L^\infty(\mathbb{R}^{2d})$.

$$\begin{aligned}(\chi_\Omega \star S) \star \check{S} &= \left(\sum_{k=1}^{\infty} \lambda_k^\Omega h_k^\Omega \otimes h_k^\Omega \right) \star \check{S} \\ &= \sum_{k=1}^{\infty} \lambda_k^\Omega (h_k^\Omega \otimes h_k^\Omega) \star \check{S} \\ &= \sum_{k=1}^{\infty} \lambda_k^\Omega Q_S(h_k^\Omega).\end{aligned}$$

Lemma

If S is a density operator and $\Omega \subset \mathbb{R}^{2d}$ a compact domain, the eigenvalues of $\chi_\Omega \star S$ satisfy $0 \leq \lambda_k^\Omega \leq 1$.

Lemma

Let $\Omega \subset \mathbb{R}^{2d}$ be a compact domain, and let $S \in \mathcal{T}^1$.

$$\mathrm{tr}(\chi_\Omega \star S) = \sum_{k=1}^{\infty} \lambda_k^\Omega = |\Omega| \mathrm{tr}(S),$$

$$\mathrm{tr}((\chi_\Omega \star S)^2) = \int_\Omega \int_\Omega \tilde{S}(z - z')\, dz\, dz'.$$

Lemma

Let S be a density operator, let $\Omega \subset \mathbb{R}^{2d}$ be a compact domain and fix $\delta \in (0,1)$. Then

$$\left|\#\{k \geq 1 : \lambda_k^\Omega > 1-\delta\} - |\Omega|\right| \leq \max\left\{\frac{1}{\delta}, \frac{1}{1-\delta}\right\} \left|\int_\Omega \int_\Omega \tilde{S}(z-z')dzdz$$

Theorem

Let S be a density operator, let $\Omega \subset \mathbb{R}^{2d}$ be a compact domain and fix $\delta \in (0,1)$. If $\{\lambda_k^{R\Omega}\}_{k\in\mathbb{N}}$ are the eigenvalues of $\chi_{R\Omega} \star S$, then

$$\frac{\#\{k : \lambda_k^{R\Omega} > 1-\delta\}}{R^{2d}|\Omega|} \to 1 \text{ as } R \to \infty.$$

Accumulated spectrogram

For any density operator S and compact domain $\Omega \subset \mathbb{R}^{2d}$, we define an associated *accumulated Cohen class distribution* by

$$\rho_\Omega^S(z) := \sum_{k=1}^{A_\Omega} Q_S(h_k^\Omega) \quad \text{for } z \in \mathbb{R}^{2d},$$

where $A_\Omega = \lceil |\Omega| \rceil$ and h_k^Ω are the eigenfunctions of $\chi_\Omega \star S$.

Note that ρ_Ω^S may also be written as a convolution of operators, since

$$\rho_\Omega^S = \sum_{k=1}^{A_\Omega} \check{S} \star (h_k^\Omega \otimes h_k^\Omega) = \check{S} \star \sum_{k=1}^{A_\Omega} (h_k^\Omega \otimes h_k^\Omega).$$

We have that $\rho_\Omega^S(z) \leq 1$.

Asymptotic convergence of accumulated Cohen's class distributions to the characteristic function of the domain.

Theorem

Let S be a density operator and $\Omega \subset \mathbb{R}^{2d}$ a compact domain. Then

$$\|\rho^S_{R\Omega}(R\,\cdot) - \chi_\Omega\|_{L^1} \to 0 \text{ as } R \to \infty.$$

The above result shows that the domain Ω is uniquely determined by $\rho^S_{R\Omega}$ as $R \to \infty$, i.e. from knowledge of S and the first $A_{R\Omega} = \lceil |R\Omega| \rceil$ eigenfunctions of $\chi_{R\Omega} \star S$ for infinitely many R.

Non-asymptotic convergence

To quantify the size of the perimeter, we will use the variation of its characteristic function χ_Ω: $|\partial\Omega| = Var(\chi_\Omega)$ for a domain $\Omega \subset \mathbb{R}^{2d}$. We say that Ω has finite perimeter if χ_Ω has bounded variation.

Definition

We also define a subset M^*_{op} of density operators by

$$M^*_{op} = \{S \in \mathcal{T}^1 : S \geq 0, \operatorname{tr}(S) = 1 \text{ and } \int_{\mathbb{R}^{2d}} \tilde{S}(z)|z|\, dz < \infty\},$$

where $|z|$ is the Euclidean norm of z, with the associated norm

$$\|S\|^2_{M^*_{op}} = \int_{\mathbb{R}^{2d}} \tilde{S}(z)|z|\, dz.$$

In terms of the Weyl symbol ϕ of $\check{S}$, that $\phi \in$ and $\int_{\mathbb{R}^{2d}} \phi * \check{\phi}(z)|z|\, dz < \infty$.

This norm lets us bound the approximation of χ_Ω by $\chi_\Omega * \tilde{S}$:

$$\|\chi_\Omega - \chi_\Omega * \tilde{S}\|_{L^1} \leq |\partial\Omega| \|S\|^2_{M^*_{op}}.$$

Lemma

Let $\Omega \subset \mathbb{R}^{2d}$ be a compact domain with finite perimeter and $S \in M^*_{op}(\mathbb{R}^d)$. If $\delta \in (0,1)$, then

$$\left| \#\{k : \lambda_k^\Omega > 1 - \delta\} - |\Omega| \right| \leq \max\left\{\frac{1}{\delta}, \frac{1}{1-\delta}\right\} \|S\|^2_{M^*} |\partial\Omega|$$

Theorem

If $S \in M^*_{op}$ and $\Omega \subset \mathbb{R}^{2d}$ is a compact domain with finite perimeter, then

$$\frac{1}{|\Omega|} \|\rho_\Omega^S - \chi_\Omega * \tilde{S}\|_{L^1} \leq \left(\frac{1}{|\Omega|} + 4\|S\|_{M^*_{op}} \sqrt{\frac{|\partial\Omega|}{|\Omega|}} \right).$$

Lemma

If $S \in M_{op}^*$ and $\Omega \subset \mathbb{R}^{2d}$ is a compact domain with finite perimeter such that $\|S\|_{M_{op}^*}^2 |\partial\Omega| \geq 1$, then for any $\delta > 0$

$$\left|\left\{z \in \mathbb{R}^{2d} : \left|\rho_\Omega^S(z) - \chi_\Omega * \tilde{S}(z)\right| > \delta\right\}\right| \lesssim \frac{1}{\delta^2} \|S\|_{M_{op}^*}^2 |\partial\Omega|.$$

Non-asymptotic convergence

If $S \in M_{op}^*$ and $\Omega \subset \mathbb{R}^{2d}$ is a compact domain with finite perimeter such that $\|S\|_{M_{op}^*}^2 |\partial\Omega| \geq 1$, then for any $\delta > 0$

$$\left|\left\{z \in \mathbb{R}^{2d} : \left|\rho_\Omega^S(z) - \chi_\Omega(z)\right| > \delta\right\}\right| \lesssim \frac{1}{\delta^2} \|S\|_{M_{op}^*}^2 |\partial\Omega|.$$

Sharp bounds

Fix $\epsilon > 0$. If $S \in M^*_{op}$ and $\Omega \subset \mathbb{R}^{2d}$ is a compact domain with finite perimeter satisfying $|\partial\Omega| \geq \epsilon$, then

$$\|\rho^S_\Omega - \chi_\Omega\|_{L^1} \leq (1/\epsilon + 2\|S\|^2_{M^*_{op}})|\partial\Omega|.$$

By considering Euclidean balls $B(z, R) = \{z' \in \mathbb{R}^{2d} : |z| < R\}$, we show that these bounds are sharp.

Sharpness

Let S be a density operator. Then there exists a constant C_S such that

$$\mathrm{tr}(\chi_{B(0,R)} \star S) - \mathrm{tr}((\chi_{B(0,R)} \star S)^2) \geq C_S R^{2d-1}, \quad \text{for } R > 1.$$

Crucial identities

Lemma

Let $\Omega \subset \mathbb{R}^{2d}$ be a compact domain, and let $S \in \mathcal{T}^1$.

$$\operatorname{tr}(\chi_\Omega \star S) = \sum_{k=1}^{\infty} \lambda_k^\Omega = |\Omega| \operatorname{tr}(S),$$

$$\operatorname{tr}((\chi_\Omega \star S)^2) = \int_\Omega \int_\Omega \tilde{S}(z - z')\, dz\, dz'.$$

Projection functional

Let S be a density operator and $\Omega \subset \mathbb{R}^{2d}$ a compact domain. Then

$$\operatorname{tr}(\chi_\Omega \star S) - \operatorname{tr}((\chi_\Omega \star S)^2) = \int_\Omega \int_{\mathbb{R}^{2d} \setminus \Omega} \tilde{S}(z - z')\, dz' dz.$$