Quantum Harmonic Analysis and its Applications

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Quantum variants of convolutions

- For $S, T \in \mathcal{T}^1$ we define the convolution of S and T

$$S * T(z) := \mathrm{tr}(S\alpha_z(\check{T})),$$

  where $\check{T} := PTP$ is the parity operator.
- For $f \in L^1(\mathbb{R})$ and $S \in \mathcal{T}^1$ we define the convolution of f and S by

$$f * S := \iint_{\mathbb{R}^2} f(y)\alpha_y(S)\, dy$$

General Moyal idenity

Let $S, T \in \mathcal{T}^1$. The function $z \mapsto \mathrm{tr}(S\alpha_z T)$ for $z \in \mathbb{R}^{2d}$ is integrable and

$$\|\mathrm{tr}(S\alpha_z \check{T})\|_{L^1} \leq \|S\|_{\mathcal{T}^1}\|T\|_{\mathcal{T}^1}.$$

Furthermore,

$$\iint_{\mathbb{R}^{2d}} \mathrm{tr}(S\alpha_z \check{T})\, dz = \mathrm{tr}(S)\mathrm{tr}(T).$$

Fourier-Wigner transform

The **Fourier- Wigner transform** $\mathcal{F}_W S$ of S is the function given by

$$\mathcal{F}_W S(z) = e^{-\pi i x \cdot \omega}\mathrm{tr}(\pi(-z)S)$$

for $z \in \mathbb{R}^{2d}$.

Mapping properties

Let $f \in L^1(\mathbb{R}^{2d})$ and $S, T \in \mathcal{T}^1$.

1. $\mathcal{F}_\sigma(S * T) = \mathcal{F}_W(S)\mathcal{F}_W(T)$.
2. $\mathcal{F}_W(f * S) = \mathcal{F}_\sigma(f)\mathcal{F}_W(S)$.

Integrated Schrödinger representation

The **integrated Schrödinger representation**, which is the map $\rho : L^1(\mathbb{R}^{2d}) \to B(L^2(\mathbb{R}^d))$ given by

$$\rho(f) = \iint_{\mathbb{R}^{2d}} f(z)e^{-\pi i x \cdot \omega}\pi(z)\, dz,$$

Tauberian Theorems

Wiener Tauberian Theorem

- $\mathrm{span}\{T_z f : z \in \mathbb{R}\}$ is norm dense in $L^1(\mathbb{R})$ if and only if $\{z \in \mathbb{R} : \widehat{f}(z) = 0\}$ is empty.
- $\mathrm{span}\{T_z f : z \in \mathbb{R}\}$ is norm dense in $L^2(\mathbb{R})$ if and only if $\{z \in \mathbb{R} : \widehat{f}(z) = 0\}$ has Lebesgue measure zero.
- $\mathrm{span}\{T_z f : z \in \mathbb{R}\}$ is a weak* dense subspace of $L^\infty(\mathbb{R})$ if and only if $\{z \in \mathbb{R} : \widehat{f}(z) = 0\}$ has dense complement.

In order to state the main results of this section, we will introduce the notion of **regularity**.

Notation

- For $1 \leq p < \infty$, we say that $g \in L^p(\mathbb{R}^{2d})$ is p-**regular** if the translates $\{T_z g : z \in \mathbb{R}^{2d}\}$ span a norm dense subspace of $L^p(\mathbb{R}^{2d})$.
- Similarly, we say that $S \in \mathcal{T}^p$ is p-**regular** if the translates $\{\alpha_z S : z \in \mathbb{R}^{2d}\}$ span a norm dense subspace of $\mathcal{T}^p$. We will often refer to 1-regularity as **regularity**.
- If $g \in L^\infty(\mathbb{R}^{2d})$ we say that g is ∞-**regular** if the translates $\{T_z g : z \in \mathbb{R}^{2d}\}$ span a weak* dense subspace of $L^\infty(\mathbb{R}^{2d})$.
- We say that $S \in B(L^2(\mathbb{R}^d))$ is ∞-**regular** if the translates $\{\alpha_z S : z \in \mathbb{R}^{2d}\}$ span a norm dense subspace of $K(L^2(\mathbb{R}^d))$.

Remark

1. It is clear that $\| \cdot \|_{B(L^2)} \leq \| \cdot \|_{\mathcal{T}^q} \leq \| \cdot \|_{\mathcal{T}^p} \leq \| \cdot \|_{\mathcal{T}^1}$ for $1 \leq p \leq q < \infty$, and that $\mathcal{T}^p$ is a dense subspace of $\mathcal{T}^q$. Thus we get that p-regularity implies q-regularity for an operator S if $p \leq q$. This is also true for $q = \infty$, since any Schatten p-class is norm dense in $K(L^2(\mathbb{R}^d))$.
2. An equivalent definition for an operator S to be ∞-regular is that the translates of S span a weak* dense subspace of $B(L^2(\mathbb{R}^d))$. We will use both of these formulations.

Wiener's Tauberian theorem

1. $f \in L^1(\mathbb{R}^{2d})$ is regular $\iff$ the set $\{z \in \mathbb{R}^{2d} : \mathcal{F}_\sigma f(z) = 0\}$ is empty.
2. $f \in L^2(\mathbb{R}^{2d})$ is 2-regular $\iff$ the set $\{z \in \mathbb{R}^{2d} : \mathcal{F}_\sigma f(z) = 0\}$ has Lebesgue measure zero.
3. $f \in L^\infty(\mathbb{R}^{2d})$ is ∞-regular $\iff$ the set $\{z \in \mathbb{R}^{2d} : \mathcal{F}_\sigma f(z) = 0\}$ has dense complement.

Remark

For $1 < p < 2$, Lev and Olevskii have shown the existence of two functions in $L^1(\mathbb{R})$ with the same set of zeros for the Fourier transform, but where one function is p-regular and the other is not. Wiener's Tauberian theorem can therefore not be extended in an obvious way to all values of $1 \leq p \leq \infty$.

Lemma

Let $S \in \mathcal{T}^1$, let $1 \leq p \leq \infty$, and let q be the conjugate exponent of p determined by $\frac{1}{p} + \frac{1}{q} = 1$. The following are equivalent:

1. S is p-regular.
2. If $f \in L^q(\mathbb{R}^{2d})$ and $f * S = 0$, then $f = 0$.
3. $\mathcal{T}^p * S$ is dense in $L^p(\mathbb{R}^{2d})$.
4. If $T \in \mathcal{T}^q$ and $T * S = 0$, then $T = 0$.
5. $L^p(\mathbb{R}^{2d}) * S$ is dense in $\mathcal{T}^p$.
6. $S * S$ is p-regular.
7. For any *regular* $T_0 \in \mathcal{T}^1$, $T_0 * S$ is p-regular.

The density in points (3) and (5) is in the p norm for $p < \infty$, and weak* density for $p = \infty$.

Lemma – ctd.

For the case $p = \infty$ we may add two further equivalent statements to the list:

(i) $K(L^2(\mathbb{R}^d)) * S$ is dense in $C_0(\mathbb{R}^{2d})$ in the $\| \cdot \|_{L^\infty}$ norm.

(ii) $C_0(\mathbb{R}^{2d}) * S$ is dense in $K(L^2(\mathbb{R}^d))$ in the operator norm $\| \cdot \|_{B(L^2)}$.

Finally, there exists a p-regular operator S for any $1 \leq p \leq \infty$.

Remark

We are now in the position to formulate one of the pillars of quantum harmonic analysis.

Werner's Tauberian Theorem

Let $S \in \mathcal{T}^1$.

1. S is regular $\iff$ the set $\{z \in \mathbb{R}^{2d} : \mathcal{F}_W S(z) = 0\}$ is empty.
2. S is 2-regular $\iff$ the set $\{z \in \mathbb{R}^{2d} : \mathcal{F}_W S(z) = 0\}$ has Lebesgue measure zero.
3. S is ∞-regular $\iff$ the set $\{z \in \mathbb{R}^{2d} : \mathcal{F}_W S(z) = 0\}$ has dense complement.

Version for pseudodifferential operators

Let $S \in \mathcal{T}^1$ be the operator on $L^2(\mathbb{R}^d)$ with twisted Weyl symbol $f \in M^1(\mathbb{R}^{2d})$.

1. S is regular $\iff$ the set $\{z \in \mathbb{R}^{2d} : f(z) = 0\}$ is empty.
2. S is 2-regular $\iff$ the set $\{z \in \mathbb{R}^{2d} : f(z) = 0\}$ has Lebesgue measure zero.
3. S is ∞-regular $\iff$ the set $\{z \in \mathbb{R}^{2d} : f(z) = 0\}$ has dense complement.

Werner Tauberian Theorem

Let $S \in \mathcal{T}^1$.

1. The set $\{z \in \mathbb{R}^{2d} : \mathcal{F}_W(S) = 0\}$ is empty.
2. If $f \in L^\infty(\mathbb{R}^{2d})$ and $f \star S = 0$, then $f = 0$.
3. $L^1(\mathbb{R}^{2d}) \star S$ is dense in $\mathcal{T}^1$.
4. If $T \in B(L^2(\mathbb{R}^d))$ and $S \star T = 0$, then $T = 0$.

Parts of the results in this section may be formulated using a notion of spectrum for a group of automorphisms on a von Neumann algebra.

Arveson spectrum – Definition

Let X be a von Neumann algebra with an automorphism group $\{U_z\}_{z\in\mathbb{R}^{2d}}$ on X, and let $x \in X$. Arveson defined the spectrum $\mathrm{sp}_U(x)$ to be the spectrum of the family of functions $\{z \mapsto \rho(U_z x) : \rho \in X_*\}$, where X_* is the predual of X considered as a subspace of the dual space of X.

Our setting

We will consider $U = \alpha$ and $X = B(L^2(\mathbb{R}^d))$, and then the predual of X is $\mathcal{T}^1$, where $T \in \mathcal{T}^1$ acts on $S \in B(L^2(\mathbb{R}^d))$ by $S \mapsto \mathrm{tr}(T^*S)$ as before. By the spectrum of a function in $f \in L^1(\mathbb{R}^{2d})$ we will mean the closed support of $\mathcal{F}_\sigma f$; with this convention the spectrum of $S \in B(L^2(\mathbb{R}^d))$ is related to the set of zeros of $\mathcal{F}_W(S)$ in a natural way.

Arveson spectrum

Let $S \in \mathcal{T}^1$. The spectrum $\mathrm{sp}_\alpha(S)$ is the closure of the complement of $\{z \in \mathbb{R}^{2d} : \mathcal{F}_W S(-z) = 0\}$.

Proof

By definition, $\mathrm{sp}_\alpha(S)$ is the spectrum of the functions $\mathrm{tr}(T^* \alpha_z S) = T^* * \check{S}(z)$, i.e. the closure of the complement of the set $Z := \{z \in \mathbb{R}^{2d} : \mathcal{F}_\sigma(T^* * \check{S}) = 0 \;\; \forall\, T \in \mathcal{T}^1\}$. We have $\mathcal{F}_\sigma(T^* * \check{S})(z) = \mathcal{F}_W(T^*)(z)\mathcal{F}_W(\check{S})(z) = \mathcal{F}_W(T^*)(z)\mathcal{F}_W(S)(-z)$, hence $\{z \in \mathbb{R}^{2d} : \mathcal{F}_W S(-z) = 0\}$ is a subset of Z. To see that $Z = \{z \in \mathbb{R}^{2d} : \mathcal{F}_W S(-z) = 0\}$, note that there exist $T \in \mathcal{T}^1$ with $\mathcal{F}_W(T^*)(z) \neq 0$ for any $z \in \mathbb{R}^{2d}$.

Remark

Hence Werner's Tauberian Theorem for $p = \infty$ yield a characterization of those $S \in \mathcal{T}^1$ where the Arveson spectrum is all of $\mathbb{R}^{2d}$.

In order to apply these results to localization operators, we pick $S = \varphi_2 \otimes \varphi_1$ for two windows $\varphi_1, \varphi_2 \in L^2(\mathbb{R}^d)$, and formulate Werner's Tauberian Theorem in the terminology of the Berezin transform and localization operators.

Lemma

Fix two windows $\varphi_1, \varphi_2 \in L^2(\mathbb{R}^d)$ for $\mathcal{A}$ and $\mathcal{B}$, let $1 \leq p \leq \infty$ and let q be the conjugate exponent of p, i.e. $\frac{1}{p} + \frac{1}{q} = 1$. The following are equivalent:

1. The operator $\varphi_2 \otimes \varphi_1$ is p-regular.
2. $\mathcal{A}$ is injective on $L^q(\mathbb{R}^{2d})$.
3. The set $\{\mathcal{B}T : T \in \mathcal{T}^p\}$ is dense in $L^p(\mathbb{R}^{2d})$.
4. $\mathcal{B}$ is injective on $\mathcal{T}^q$.
5. The set $\{\mathcal{A}f : f \in L^p(\mathbb{R}^{2d})\}$ is dense in $\mathcal{T}^p$.
6. $\mathcal{B}(\varphi_2 \otimes \varphi_1)$ is p-regular.
7. For any *regular* $T_0 \in \mathcal{T}^1$, $\mathcal{B}T$ is p-regular.

The density in points (1), (3) and (5) is in the p norm for $p < \infty$, and weak* density for $p = \infty$.

If we apply Werner's Tauberian theorem to localization operators, we obtain a characterization of those windows $\varphi_1, \varphi_2 \in L^2(\mathbb{R}^d)$ that give localization operators $\mathcal{A}^{\varphi_1,\varphi_2}$ with dense range in $\mathcal{T}^1$, $\mathcal{T}^2$ and $B(L^2(\mathbb{R}^d))$. This improves results by Bayer and Gröchenig.

Tauberian theorems for localization operators

Fix two windows $\varphi_1, \varphi_2 \in L^2(\mathbb{R}^d)$.

1. The set $\{\mathcal{A}_f^{\varphi_1,\varphi_2} : f \in L^1(\mathbb{R}^{2d})\}$ is dense in $\mathcal{T}^1 \iff$ the set $\{z \in \mathbb{R}^{2d} : A(\varphi_2, \varphi_1)(z) = 0\}$ is empty.
2. The set $\{\mathcal{A}_f^{\varphi_1,\varphi_2} : f \in L^2(\mathbb{R}^{2d})\}$ is dense in $\mathcal{T}^2 \iff$ the set $\{z \in \mathbb{R}^{2d} : A(\varphi_2, \varphi_1)(z) = 0\}$ has zero Lebesgue measure.
3. The set $\{\mathcal{A}_f^{\varphi_1,\varphi_2} : f \in L^\infty(\mathbb{R}^{2d})\}$ is weak* dense in $B(L^2(\mathbb{R}^d))$ $\iff$ the set $\{z \in \mathbb{R}^{2d} : A(\varphi_2, \varphi_1)(z) = 0\}$ has dense complement.

Schwartz operators

(1) Let $S, T \in \mathfrak{S}$, $A \in \mathfrak{S}'$, $f \in \mathcal{S}(\mathbb{R}^{2d})$ and $\phi \in \mathcal{S}'(\mathbb{R}^{2d})$. The following convolutions may be defined:

$$S \star T \in \mathcal{S}(\mathbb{R}^{2d}) \qquad f \star S \in \mathfrak{S}$$
$$S \star A \in \mathcal{S}'(\mathbb{R}^{2d}) \qquad \phi \star S \in \mathfrak{S}'$$
$$f \star A \in \mathfrak{S}'.$$

(2) The Fourier-Wigner transform may be extended to a topological isomorphism $\mathcal{F}_W : \mathfrak{S}' \to \mathcal{S}'(\mathbb{R}^{2d})$.

(3) The relations $\mathcal{F}_\sigma(S \star T) = \mathcal{F}_W(S)\mathcal{F}_W(T)$ and $\mathcal{F}_W(f \star S) = \mathcal{F}_\sigma(f)\mathcal{F}_W(S)$ still hold for operators S, T and a function f whenever the convolutions are defined by part (1).

Remark

By the Schwartz kernel theorem, we know that we may identify $\mathfrak{S}'$ with the continuous operators from $\mathcal{S}(\mathbb{R}^{2d})$ to $\mathcal{S}'(\mathbb{R}^{2d})$.

Uncertainty principle

Let $S \in \mathcal{T}^1$ and let $\Omega \subset \mathbb{R}^{2d}$ with $\mu(\Omega) < \infty$ and assume that

$$\iint_\Omega |\mathcal{F}_W(S)(z)|^2 \, dz \geq 1 - \epsilon$$

for some $\epsilon \geq 0$. For any $p > 2$ we then have

$$\mu(\Omega) \geq \frac{(1-\epsilon)^{p/(p-2)} \left(\frac{p}{2}\right)^{2d/(p-2)}}{\|S\|_{\mathcal{T}^1}^{2p/(p-2)}}.$$

In particular, for $p = 4$ we obtain

$$\mu(\Omega) \geq \frac{(1-\epsilon)^2 2^d}{\|S\|_{\mathcal{T}^1}^4}.$$

One interpretation of this uncertainty principle is that a well-concentrated spreading function comes at the cost of a large trace class norm.

Among the localization operators $\mathcal{A}_f^{\varphi_1,\varphi_2}$, those of the form $\mathcal{A}_\Omega^{\varphi}$ for some measurable $\Omega \subset \mathbb{R}^{2d}$ have a special interpretation:
if $\psi \in L^2(\mathbb{R}^d)$, the signal $\mathcal{A}_\Omega^{\varphi}\psi$ is interpreted as the part of ψ "living on" Ω [**Cordero:2003**], which explains the "localization" terminology.

Mixed-state localization operators

We define a **mixed-state localization operator** to be an operator H_Ω of the form

$$H_\Omega = \chi_\Omega \star S$$

where $\Omega \subset \mathbb{R}^{2d}$ is a measurable subset and S is a positive trace class operator with $\mathrm{tr}(S) = 1$.

Question

Given a mixed-state localization operator $\chi_\Omega \star S$, one might ask whether it is possible to recover information about the domain Ω from the operator $\chi_\Omega \star S$. The next proposition shows that the measure of Ω may be calculated from the eigenvalues $\chi_\Omega \star S$.

Proposition

Let $\Omega \subset \mathbb{R}^{2d}$ be a subset of finite Lebesgue measure, and let $S \in \mathcal{T}^1$ be a positive operator with $\mathrm{tr}(S) = 1$. Then

1. $\mathrm{tr}(\chi_\Omega \star S) = \mu(\Omega)$, where μ is Lebesgue measure.
2. If $\{\lambda_i\}_{i=1}^N$ are the eigenvalues of $\chi_\Omega \star S$ counted with algebraic multiplicity, then

$$\sum_{i=1}^{N} \lambda_i = \mu(\Omega).$$

Remark

1. The proof of this proposition would work equally well if χ_Ω is replaced by any $f \in L^1(\mathbb{R}^{2d})$, as long as $\mu(\Omega)$ is replaced by $\iint_{\mathbb{R}^{2d}} f(z)\, dz$.
2. This result holds in particular for the localization operators $\mathcal{A}_f^\varphi$ by picking $S = \varphi \otimes \varphi$. In this context it is well-known, see for instance.

In recent years the question of obtaining the domain Ω from the localization operator $\mathcal{A}_\Omega^\varphi$ has received some attention. We will consider th possibility of such reconstruction for the mixed-state localization operators.

Question

If $S \in \mathcal{T}^1$, when is the domain $\Omega \subset \mathbb{R}^{2d}$ uniquely determined by the mixed-state localization operator $\chi_\Omega \star S$, up to sets of Lebesgue measure zero?
Since the localization operators $\mathcal{A}_\Omega^\varphi$ form a subset of the mixed-state localization operators, our results will also be applicable to such operators.

Theorem

1. If $S \in \mathcal{T}^1$ is such that the set $\{z \in \mathbb{R}^{2d} : \mathcal{F}_W(S)(z) = 0\}$ has dense complement in $\mathbb{R}^{2d}$, then any $\Omega \subset \mathbb{R}^{2d}$ with finite Lebesgue measure is uniquely determined by the operator $\chi_\Omega \star S$, up to Lebesgue measure zero.
2. If $\varphi_1, \varphi_2 \in L^2(\mathbb{R}^d)$ are windows such that the set $\{z \in \mathbb{R}^{2d} : A(\varphi_2, \varphi_1)(z) = 0\}$ has dense complement in $\mathbb{R}^{2d}$, then any $\Omega \subset \mathbb{R}^{2d}$ with finite Lebesgue measure is uniquely determined by the operator $\mathcal{A}_\Omega^{\varphi_1,\varphi_2}$, up to Lebesgue measure zero.

Unbounded case

1. If $S \in \mathcal{T}^1$ is such that the set $\{z \in \mathbb{R}^{2d} : \mathcal{F}_W(S)(z) = 0\}$ is empty, then any measurable $\Omega \subset \mathbb{R}^{2d}$ is uniquely determined by the operator $\chi_\Omega \star S$, up to Lebesgue measure zero.
2. If $\varphi_1, \varphi_2 \in L^2(\mathbb{R}^d)$ are windows such that the set $\{z \in \mathbb{R}^{2d} : A(\varphi_2, \varphi_1)(z) = 0\}$ is empty, then any measurable $\Omega \subset \mathbb{R}^{2d}$ is uniquely determined by the operator $\mathcal{A}_\Omega^{\varphi_1,\varphi_2}$, up to Lebesgue measure zero.

Since an unbounded set Ω may have infinite Lebesgue measure, we will not be able to use that $\chi_\Omega \in L^1(\mathbb{R}^{2d})$ as we did in the proof of the previous corollary. We need to consider χ_Ω as an element of $L^\infty(\mathbb{R}^{2d})$. This leads to a stronger condition on the set of zeros of the Fourier-Wigner transform.

Cohen class

A quadratic time-frequency distribution Q is said to be of *Cohen's class* if Q is given by

$$Q(\psi) = Q_\phi(\psi) := W(\psi,\psi) * \phi$$

for some $\phi \in \mathcal{S}'(\mathbb{R}^{2d})$.

The class of functions ψ to which we may apply Q_ϕ clearly depends on the distribution ϕ. The Wigner distribution is obtained by picking $\phi = \delta_0$, where δ_0 is Dirac's delta distribution centered at 0.

Lemma

Let Q be a quadratic time-frequency distribution satisfying

1. $Q(\pi(z)\psi) = T_z(Q(\psi))$,
2. $|Q(\psi_1,\psi_2)(0)| \leq \|\psi_1\|_2 \|\psi_2\|_2$,

for all $z \in \mathbb{R}^{2d}$ and $\psi_1, \psi_2 \in L^2(\mathbb{R}^d)$. Then $Q(\psi) = W(\psi,\psi) * \phi$ for some $\phi \in \mathcal{S}'(\mathbb{R}^{2d})$.

We give a characterization of Cohen's class as convolutions with a fixed operator.

Cohen class characterization

For $\phi \in \mathcal{S}'(\mathbb{R}^{2d})$, the associated Cohen's class distribution Q_ϕ is given by

$$Q_\phi(\psi) = (\psi \otimes \psi) \star L_\phi \ \text{ for } \psi \in \mathcal{S}(\mathbb{R}^d), \tag{1}$$

where L_ϕ is the Weyl transform of ϕ.
Conversely, any operator $A \in \mathfrak{S}'$ determines a Cohen's class distribution by

$$Q_A(\psi) := (\psi \otimes \psi) \star \check{A} \ \text{ for } \psi \in \mathcal{S}(\mathbb{R}^d).$$

Comments

1. This proposition shows that any shift-invariant, $Q(\pi(z)\psi) = T_z(Q(\psi))$ for $z \in \mathbb{R}^{2d}$ and $\psi \in L^2(\mathbb{R}^d)$, weakly continuous quadratic time-frequency distribution is given by a convolution with a fixed operator on $L^2(\mathbb{R}^d)$.
2. Hence, any Cohen's class distribution may be described by either a distribution $\phi \in \mathcal{S}'(\mathbb{R}^{2d})$ or by an operator $A \in \mathfrak{S}'$, where
$$Q_\phi = Q_A \text{ if } L_\phi = \check{A}.$$
3. We have defined Q_A in terms of $\check{A}$ to simplify formulas, note that A and $\check{A}$ share all relevant properties, such as positivity, trace and membership of Schatten classes.
4. Using that $Q_\phi(\psi) = (\psi \otimes \psi) \star L_\phi$, we may apply the theory of convolutions of operators to deduce some simple results on Cohen's class distributions.

Examples

1. The Wigner distribution $Q_\phi(\psi) = W(\psi, \psi)$ is given by $\phi = \delta_0$. $W(\psi, \psi)$ is also given by

$$W(\psi, \psi) = (\psi \otimes \psi) \star L_{\delta_0}$$

   for $\psi \in \mathcal{S}(\mathbb{R}^d)$. By a result of Grossmann, $L_{\delta_0} = 2^d P$, where P is the parity operator.

2. Fix a window $\varphi \in L^2(\mathbb{R}^d)$ and consider the operator $S = \varphi \otimes \varphi$. Then $\check{S} = \check{\varphi} \otimes \check{\varphi}$ defines a Cohen's class distribution Q_S by

$$Q_S(\psi) = (\psi \otimes \psi) \star (\check{\varphi} \otimes \check{\varphi}) = |V_\varphi \psi|^2.$$

   This Cohen's class distribution is therefore the spectrogram. The corresponding function ϕ, i.e. the Weyl symbol of $\check{\varphi} \otimes \check{\varphi}$, is the Wigner distribution $W(\check{\varphi}, \check{\varphi})$.

Proposition

Fix $1 \leq p \leq \infty$. Consider a Cohen's class distribution Q_ϕ for $\phi \in \mathcal{S}'(\mathbb{R}^{2d})$. Let L_ϕ be the Weyl transform of ϕ.
If $L_\phi \in \mathcal{T}^p$, then $Q_\phi(\psi)$ is well-defined for any $\psi \in L^2(\mathbb{R}^d)$ and $Q_\phi(\psi) \in L^p(\mathbb{R}^{2d})$ with $\|Q(\psi)\|_{L^p} \leq \|\psi\|_{L^2}^2 \|S\|_{\mathcal{T}^p}$.
In particular, if $L_\phi \in B(L^2(\mathbb{R}^d))$, then $Q_\phi(\psi) \in L^\infty(\mathbb{R}^{2d})$ with $\|Q(\psi)\|_{L^\infty} \leq \|\psi\|_{L^2}^2 \|S\|_{B(L^2(\mathbb{R}^d))}$.

Remark

By Pool's Theorem, the condition that $L_\phi \in \mathcal{T}^2$ is equivalent to $\phi \in L^2(\mathbb{R}^{2d})$. Unfortunately there is no equally simple characterization of those ϕ such that $L_\phi \in \mathcal{T}^1$ or $L_\phi \in B(L^2(\mathbb{R}^d))$.

We say that a Cohen's class distribution Q_ϕ is **positive** if $Q_\phi(\psi)(z) \geq 0$ for all $z \in \mathbb{R}^{2d}$ and ψ in the domain of Q_ϕ.

Positivity

Let Q_ϕ be a Cohen's class distribution such that the Weyl transform L_ϕ is bounded on $L^2(\mathbb{R}^d)$. Then Q_ϕ is positive if and only if L_ϕ is a positive operator.

we may equivalently ask which conditions ϕ must satisfy to ensure that the Weyl transform L_ϕ is a positive operator.
This question is of interest in quantum mechanics, and providing a general answer has turned out to be difficult. The so-called KLM conditions due to Kastler, Loupias and Miracle-Sole.

By using the connection between Cohen's class and convolutions of operators we obtain a weak uncertainty principle for Cohen's class distributions. The result is modeled on uncertainty principles for the spectrogram and Wigner distribution.

Proposition

Let $S \in B(L^2(\mathbb{R}^d))$ and let Q_S be the Cohen's class distribution determined by $Q_S(\psi) = (\psi \otimes \psi) \star S$ for $\psi \in L^2(\mathbb{R}^d)$. If $\Omega \subset \mathbb{R}^{2d}$ is a measurable subset such that

$$\iint_\Omega |Q_S(\psi)| \, dz \geq (1-\epsilon)\|S\|_{B(L^2(\mathbb{R}^d))}$$

for some $\psi \in L^2(\mathbb{R}^d)$ with $\|\psi\|_{L^2} = 1$ and $\epsilon \geq 0$, then

$$\mu(\Omega) \geq 1 - \epsilon.$$