Sobolev classes of functions valued in a monotone family of Banach spaces

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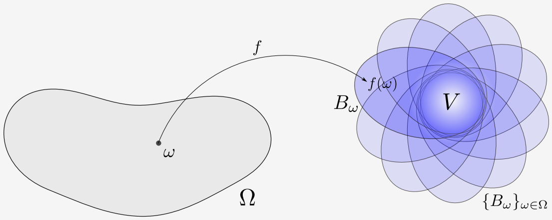
(Ω, μ), V, $\{\|\cdot\|_\omega\}_{\omega\in\Omega}$ – semi-norms, B_ω is a completion $V/\ker(\|\cdot\|_\omega)$



$$f : \Omega \to \bigcup_{\omega\in\Omega} B_\omega, \quad f(\omega) \in B_\omega$$

L_p-direct integral of Banach spaces

$$f \in L^p(\Omega, \{B_\omega\}) = \left(\int_\Omega^\oplus W_\omega \, d\mu(\omega)\right)_{L_p} \Leftrightarrow \left(\int_\Omega \|f(\omega)\|_{B_\omega}^p \, d\mu(\omega)\right)^{\frac{1}{p}} < \infty$$

R. Haydon, M. Levy, Y. Raynaud, *Randomly normed spaces*, Hermann, Paris, 1991.

How to define the sobolev space $W^{1,p}$?

Now $\Omega = I \subset \mathbb{R}$ is an interval, V – v. sp. equipped with $\{\|\cdot\|_t\}_{t\in I}$
1) For every $v \in V$ function $t \mapsto \|v\|_t$ is measurable;
2) $\|v\|_{t_1} \geq \|v\|_{t_2}$ whenever $t_1 \leq t_2$.

Say $f \in W^{1,p}(I, \{X_t\})$ if $f \in L^p(I, \{X_t\})$ and there exists a function $g \in L^p(I, \mathbb{R})$ such that

$$\|f(t) - f(t_0)\|_t \leq \int_{t_0}^{t} g(s)\, ds.$$

What can we say about function g?

Is it $|f'(t)|$?

Calculus of $\{X_t\}$-valued functions

- There are 'embedding' operators $P(t_1, t_2) : X_{t_1} \to X_{t_2}$.
- If $x_1 \in X_{t_1}$ and $x_2 \in X_{t_2}$ define

$$x_1 + x_2 := P(t_1, t_2)x_1 + x_2$$

- $P(t_1, t_2)P(t_2, t_3) = P(t_1, t_3)$

$$\bigcup_{t \in I} X_t \qquad \text{– vector space (but not normed)}$$

$$\lim_{\omega \to \omega_0} f(\omega) = \xi \in X_{t_0} \ \text{ if } \ \|f(\omega) - \xi\|_{\omega_0 \vee \omega} \to 0.$$

Calculus of $\{X_t\}$-valued functions. Integral

- **locally integrable section**: for every compact there exist a sequence of simple sections s. t.

$$\int_I \|\chi_J(t) \cdot f(t) - s_k(t)\|_t \, dt \to 0 \quad \text{for } k \to \infty.$$

- **local Bochner integral**: $\int_J f \, dt := \lim\limits_{k\to\infty} \int_I s_k(t)\, dt = x \in X_{t^*}$, where $t^* = \sup J$.
- f is a locally integrable section $\Leftrightarrow \|f(t)\|_t \in L^1_{loc}(I)$

Weak derivative

Let $f \in L^1_{loc}(I, \{X_t\})$. A function $h \in L^1_{loc}(I, \{X_t\})$ is a weak derivative of f if for any $\varphi \in C_0^\infty(I, \mathbb{R})$

$$\left\| \int_I \varphi'(t) f(t)\, dt + \int_I \varphi(t) h(t)\, dt \right\|_{t^*} = 0,$$

where $t^* = \sup\{\operatorname{supp} \varphi(t)\}$.

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Theorem

$$\|u\|_{L^p} + \|u'\|_{L^p} < \infty \Leftrightarrow u \in L^p(I, \{X_t\}) \text{ and } \|u(\omega) - u(\omega_0)\|_\omega \leq \int\limits_{\omega_0}^{\omega} g ds.$$

! if $u \in W^{1,p} \Rightarrow$ weak derivative u' exists and it is a section

Answering our question: $g(t) = \|u'(t)\|_t$

Theorem (the approach of Yu. G. Reshetnyak)

If $u : I \to \{X_t\}$ *is a measurable section and :*
(A) for any $v \in V$ *the function* $\psi_v(t) = \|u(t) - v\|_t \in W^{1,p}(I, \mathbb{R})$,
(B) the family of derivatives $\{\psi'_v(t)\}_{v \in V}$ *has a majorant* $\psi' \in L^p(I, \mathbb{R})$,
then $u \in W^{1,p}(I, \{X_t\})$.

The converse does not hold in general.

Applications

- PDE in domains that change in time

$$\dot{u}(t) + A(t)u(t) = f(t).$$

- Transport in a heterogeneous porous medium.

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Thank you for attention!