UiO : Department of Mathematics
University of Oslo

Hilbert C*-modules in harmonic analysis

Nordfjordeid 2019

Ulrik Enstad

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Let A be a unital C*-algebra. A left Hilbert A-module is a left A-module $\mathcal{E}$ together with a map ${}_\bullet\langle \cdot, \cdot \rangle : \mathcal{E} \times \mathcal{E} \to A$ such that:

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1. For all $\xi, \eta, \gamma \in \mathcal{E}$ and $a, b \in A$ we have that

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3. For all $\xi \in \mathcal{E}$ we have that ${}_\bullet\langle \xi, \xi \rangle \geq 0$, and ${}_\bullet\langle \xi, \xi \rangle = 0$ implies $\xi = 0$.
4. $\mathcal{E}$ is complete with respect to the norm $\|\xi\|_{\mathcal{E}} := \| {}_\bullet\langle \xi, \xi \rangle \|_A^{1/2}$.

Examples of Hilbert C*-modules

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- The left A-module A^k becomes a left Hilbert A-module with the inner product

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- For a countable index set J, define

$$ \ell^2(J, A) = \{ (a_j)_{j \in J} \subseteq A : \sum_{j \in J} a_j a_j^* \text{ converges in } A \}. $$

This becomes a left Hilbert A-module with respect to

$$ {}_\bullet\langle (a_j), (b_j) \rangle = \sum_{j \in J} a_j b_j^* . $$

Sections of vector bundles

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$$(f \cdot s)(x) = f(x)s(x)$$
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Theorem (Serre–Swan)

If $\mathcal{E}$ is a finitely generated Hilbert $C(X)$-module, then there exists a unique Hermitian vector bundle $E \to X$ such that $\mathcal{E} \cong \Gamma(E)$.

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$$\overline{\left\{ \sum_{j;\text{finite}} a_j s_j : a_j \in A, s_j \in \mathcal{E} \right\}} = \mathcal{E}.$$

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- Does any countably generated Hilbert C^*-module admit a countable “orthonormal basis”?
- No!

Frames

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Definition (Frank–Larson)

Let J be a countable index set. A sequence $(\eta_j)_{j\in J} \subseteq \mathcal{E}$ is called a *frame* if there exists $C, D > 0$ such that

$$C {}_\bullet\langle \xi, \xi \rangle \leq \sum_{j \in J} {}_\bullet\langle \xi, \eta_j \rangle \, {}_\bullet\langle \xi, \eta_j \rangle^* \leq D {}_\bullet\langle \xi, \xi \rangle$$

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- The frame is *normalized tight* if one can choose $C = D = 1$.

Dual frames

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If $(\eta_j)_{j\in J}$ is a frame in $\mathcal{E}$, there exists another frame $(\gamma_j)_{j\in J}$ such that

$$\xi = \sum_{j\in J} {}_\bullet\langle \xi, \eta_j \rangle \gamma_j = \sum_{j\in J} {}_\bullet\langle xi, \gamma_j \rangle \eta_j$$

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Theorem (Frank–Larson)

If $\mathcal{E}$ is a countably generated Hilbert A-module, then it admits a (countable) frame.

Properties of frames

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- A finite set $\{\eta_1, \ldots, \eta_k\} \subseteq \mathcal{E}$ is a frame if and only if it is an (algebraic) generating set.

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- In particular, if E has rank k, a frame of k elements in $\Gamma(E)$ is exactly a global trivialization of E.

Projective modules

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- If $(\eta_1, \ldots, \eta_k)$ is a frame for $\mathcal{E}$, then the matrix

$$P = ({}_{\bullet}\langle \eta_i, \eta_j \rangle)_{i,j=1}^{k}$$

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- In particular, the K-theory class of $\mathcal{E}$ is represented by P.
- If τ is a trace on A, then the induced trace $\tilde{\tau} \colon K_0(A) \to \mathbb{C}$

$$\tilde{\tau}([\mathcal{E}]) = \sum_{j=1}^k \tau({}_\bullet\langle \eta_j, \eta_j \rangle).$$

Gabor frames

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- Given $\theta \in \mathbb{R} \setminus \{0\}$, define operators $T_\theta, M \colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by

$$T_\theta \xi(t) = \xi(t - \theta)$$
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- These two operators satisfy

$$MT_\theta = e^{2\pi i \theta} T_\theta M.$$

Problem

Given $\eta_1, \dots, \eta_k \in L^2(\mathbb{R})$ *and* $\theta \in \mathbb{R} \setminus \{0\}$, *when is the set*

$$\{M^n T_\theta^m \eta_j : 1 \leq j \leq k, m, n \in \mathbb{Z}\}$$

an orthonormal basis for $L^2(\mathbb{R})$, *or more generally a frame for* $L^2(\mathbb{R})$*?*

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A Balian–Low Theorem (Battle)

If $\eta \in \mathcal{S}(\mathbb{R})$, then

$$\{M^n T_\theta^m \eta : m, n \in \mathbb{Z}\}$$

cannot be an orthonormal basis for $L^2(\mathbb{R})$.

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- A_θ is equipped with a trace $\tau \colon A_\theta \to \mathbb{C}$ given by

$$\tau\left(\sum_{m,n} a_{m,n} M^n T_\theta^m\right) = a_{0,0}.$$

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- for $a = \sum_{m,n} a_{m,n} M^n T_\theta^n \in A_\theta$ and $\xi \in \mathcal{S}(\mathbb{R})$, define

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- We call the resulting completed left Hilbert A_θ-module a *Heisenberg module* and denote it by $\mathcal{E}_\theta$.

Theorem (Luef)

A set $\{\eta_1, \ldots, \eta_k\}$ of functions in $\mathcal{S}(\mathbb{R})$ is a (normalized tight) frame for $\mathcal{E}_\theta$ if and only if

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- If in addition $\{M^n T_\theta^m \eta_j : 1 \leq j \leq k, m, n \in \mathbb{Z}\}$ is an orthonormal basis, then

$$\tilde{\tau}([E_\theta]) = k.$$

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- By the Serre–Swan theorem, there exists a unique vector bundle $E \to \mathbb{T}^2$ such that

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- One can show that this is a line bundle with Chern class -1.

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