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Homogeneous 8-manifolds admitting invariant $\mathrm{Spin}(7)$-structures

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Plan of the talk

Part A)

• General (open) problem: Classify 8-dimensional (almost effective) homogeneous manifolds admitting an *invariant* $\mathrm{Spin}(7)$-*structure*

• Simplify: *Compact* case ⇒ simply connected case (still. . .complicated)

Tools? ⇒ Homogeneous spaces, Topology, Representation theory

Our aim: A Short Description of Methodology

Part B)

⇒ Which is the type **type** of a generic invariant $\mathrm{Spin}(7)$-structure Φ?

⇒ Connection (with skew-torsion) preserving such a structure (characteristic connection)

Some small history... of $\mathrm{Spin}(7)$

$\Rightarrow$ $\mathrm{Spin}(7)$ is an exceptional holonomy group $\Rightarrow$ $\mathrm{Spin}(7)$-manifolds (*Berger* (1955))

- There are just two (pure) classes of *non-integrable* $\mathrm{Spin}(7)$-structures: (*Fernández* (1986))

  balanced and locally conformally parallel (l.c.p)

$\longrightarrow$ Complete non-compact examples of $\mathrm{Spin}(7)$-manifolds:

- *Bryant & Salamon*, (1987 – 1989) $\Rightarrow$ First examples
- *L. Foscolo* (2019) $\Rightarrow$ arXiv:1901.04074 (AC orbitbolds with G_2-holonomy)

$\longrightarrow$ Compact examples:

- *Joyce* (1996), *Kovalev* (2003) $\Rightarrow$ Examples of compact $\mathrm{Spin}(7)$-manifolds
- *Fernández* $\Rightarrow$ Examples of compact balanced $\mathrm{Spin}(7)$-structures
- *Ivanov, Parton, and Piccinni* (2006) $\Rightarrow$ Structure of compact l.c.p $\mathrm{Spin}(7)$-structures

$\longrightarrow$ *L. Martín Merchán* (2019) $\Rightarrow$ Spinorial classification of Spin(7) structures (Annali della Scuola Norm. Super. di Pissa 2019)

- Exists unique characteristic connection preserving a given $\mathrm{Spin}(7)$-structure $\Rightarrow$ *Ivanov* (2004)

Preliminaries

- Consider $\mathbb{R}^7$ with standard basis $\{e_1, \ldots, e_7\}$, dual $\{e^1, \ldots, e^7\}$. Set $e^{ijk} = e^i \wedge e^j \wedge e^k$, etc. View $\mathbb{R}^8$ as $\mathbb{R}^8 = \mathbb{R}e_0 \oplus \mathbb{R}^7$ with standard basis $\{e_0, \ldots, e_8\}$.

- (Reichel 1907/Schouten 1931) The Lie group G_2 can be defined as the stabilizer $G_\omega = \{A \in \mathrm{GL}(7, \mathbb{R}) : \omega = A^*\omega\}$ of a (positive) stable 3-form on $\mathbb{R}^7$
$$\omega := e^{123} + e^{145} - e^{167} + e^{246} + e^{257} - e^{356} + e^{347}$$

- A G_2-structure on a manifold M^7 is a reduction of the structure group $\mathrm{GL}(7, \mathbb{R})$ of the frame bundle to G_2.

Theorem. A smooth manifold M^7 admits a G_2 structure if and only if it is **orientable** and **spin**.

- Since $\mathrm{G}_2 \subset \mathrm{SO}(7)$, any G_2- structures defines an orientation (Hodge star $*$) and a Riemannian metric $g = g_\omega$.

- The Lie group $\mathrm{Spin}(7)$ can be defined as the stabilizer $G_\Phi = \{A \in \mathrm{GL}(8, \mathbb{R}) : \Phi = A^*\Phi\}$ of the following 4-form Φ on $\mathbb{R}^8$
$$\Phi := e^0 \wedge \omega + *\omega$$

- A $\mathrm{Spin}(7)$-structure on a manifold M^8 is a reduction of the structure group $\mathrm{GL}(8, \mathbb{R})$ of the frame bundle to $\mathrm{Spin}(7)$.

Theorem. A **spin** manifold M^8 admits a $\mathrm{Spin}(7)$-structure if and only if $p_1^2(M) - 4p_2(M) + 8\chi(M) = 0$.

Another difference:

$\Rightarrow$ ω is a **stable** 3-form, i.e. $\dim \mathrm{GL}(7,\mathbb{R}) - \dim G_\omega = 49 - 14 = \dim \Lambda^3(\mathbb{R}^7)^*$

$\Rightarrow$ In contrast, $\dim \mathrm{GL}(8,\mathbb{R}) - \dim G_\Phi = 64 - 21 < 70 = \dim \Lambda^4(\mathbb{R}^8)^*$

- Thus, the space of $\mathrm{Spin}(7)$-structures on M^8 is *not* an open set!

$\Longrightarrow$ Since $\mathrm{Spin}(7) \subset \mathrm{SO}(8)$, any $\mathrm{Spin}(7)$-structure defines an orientation (Hodge star $\star$) and a Riemannian metric $h = h_\Phi$.

Types of non-integrable $\mathrm{Spin}(7)$-structures - *Fernández* Classification

- $\nabla^h \Phi$ carries the same information with the 5-form $d\Phi \Rightarrow$ *Intrinsic torsion*

- Lee form ϑ (*Cabrera*)

$$\vartheta := -\frac{1}{7} \star (\star \mathrm{d}\Phi \wedge \Phi) = \frac{1}{7} \star (\delta\Phi \wedge \Phi),$$

$\rightsquigarrow$

Type $\mathscr{W}_1$	balanced	$\vartheta = 0$
Type $\mathscr{W}_2$	l.c.p	$\mathrm{d}\Phi = \vartheta \wedge \Phi$

Part A) Homogeneous $\mathrm{Spin}(7)$-structures

Def. A $\mathrm{Spin}(7)$-structure Φ on an 8-dimensional manifold M is called *homogeneous* or *invariant* if there exists a connected Lie group G acting transitively and almost effectively on M, preserving the 4-form Φ.

$\Longrightarrow$ $M \simeq G/H$

$\Longrightarrow$ $\Phi \in (\Lambda^4 T^* M)^G$ is a G-invariant 4-form stabilized by $\mathrm{Spin}(7)$

$\Longrightarrow$ isotropy representation $\chi : H \to \mathrm{GL}(T_o M)$, $\chi(H) \subset \mathrm{GL}(T_o M)$ is a subgroup of $\mathrm{Spin}(7)$

- We are interested on **compact examples**. Thus assume from now on that

G is a compact Lie group, $H \subset G$ closed subgroup

- reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, $\mathfrak{m} \simeq T_o M^8$
- Identify the G-invariant 4-form Φ with an $\mathrm{Ad}(H)$-invariant 4-form on $\mathfrak{m}$, $\Phi \in (\Lambda^4 \mathfrak{m}^*)^H$

$\longrightarrow$ $\mathfrak{h} \simeq \chi_*(\mathfrak{h})$ must be a Lie subalgebra of $\mathfrak{spin}(7) \subset \mathfrak{gl}(\mathfrak{m})$, thus $\mathrm{rk}\mathfrak{h} \leqslant 3$.

A large class of compact examples - Interplay of G_2- and $\mathrm{Spin}(7)$-structures

• Fix a compact homogeneous 7-manifold $N = L/K$ with an invariant G_2-structure $\omega \in (\Lambda^3 T^* N)^L$. This induces an invariant Spin_7-structure on

$$M = \frac{L}{K} \times \mathrm{S}^1 = \frac{L}{K} \times \mathrm{U}(1), \quad \Phi := \eta \wedge \omega + *\omega \in (\Lambda^4 T^* M)^{L \times \mathrm{U}(1)}$$

for some non-trivial invariant 1-form η on $\mathrm{S}^1 = \mathrm{U}(1)$.

• *F. Cabrera* (1995) Any principal fibre bundle with one dimensional fibre over a 7-manifold endowed with a G_2-structure admits at least two $\mathrm{Spin}(7)$-structures.

• Conversely, every invariant Spin_7-structure Φ on the 8-manifold $M = L/K \times \mathrm{U}(1)$ induces an invariant G_2-structure $\omega = e_0 \lrcorner \Phi$ on $N^7 = L/K$.

• Classification of homogeneous manifolds with such invariant $\mathrm{Spin}(7)$-structures $\Rightarrow$ Based on the **classification of compact homogeneous spaces admitting an invariant G_2-structure**:

- *F. Reidegeld* (2010)
- *H. Van Lê and M. Munir* (2012)

Remark: The type of the invariant $\mathrm{Spin}(7)$-structure on $M^8 = N^7 \times \mathrm{S}^1$ is easily controlled by the type of the invariant G_2-structure on N^7.

The simply connected case - Our classification

Step A – The canonical presentation

- Let M be a compact, simply connected homogeneous space and let G' be a connected Lie group acting transitively and almost effectively on it.
- Starting from a presentation $M = G'/H'$ there is always a presentation of the form $M = G/H$, where G is a compact, connected, simply connected, semisimple Lie group and $H \subset G$ is a connected closed subgroup (see $\mathscr{B}\ddot{o}\mathscr{h}\mathscr{m}$)

- We can restrict our attention to such presentations.

$\Rightarrow$ Classification of compact simply-connected homogeneous spaces $M^n = G/H, n \leqslant 9$ ($\mathscr{S}.\ \mathscr{K}\ell a u s s$ (1998))

$\Rightarrow$ Specify the canonical presentation for the non-symmetric case

Table 1 - Compact simply connected almost effective non-symmetric homogeneous 8-manifolds.

	M^8	$\simeq G/H$
1)	$\mathrm{SU}(3)$	$\dfrac{\mathrm{SU}(3)}{\{e\}}$
2)	$\mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^2$	$C_{k,\ell,m} = \dfrac{\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2)}{\mathrm{U}(1)^{k,\ell,m}} \quad (\gcd(k, \ell, m) = 1)$
3)	$\mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^2$	$\dfrac{\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2)}{\Delta\,\mathrm{SU}(2)} \times \dfrac{\mathrm{SU}(2)}{\mathrm{U}(1)}$
4)	$\mathrm{S}^5_{V^4 \oplus \mathbb{R}} \times \mathrm{S}^3$	$\dfrac{\mathrm{SU}(3)}{\mathrm{SU}(2)} \times \mathrm{SU}(2)$
5)	$\mathrm{Sp}(2)$-full flag mnfd	$\dfrac{\mathrm{Sp}(2)}{\mathrm{T}^2_{\mathrm{max}}}$
6)	$\mathbb{F}^3 \times \mathrm{S}^2$	$\dfrac{\mathrm{SU}(3)}{\mathrm{T}^2_{\mathrm{max}}} \times \dfrac{\mathrm{SU}(2)}{\mathrm{U}(1)}$
7)	$\mathbb{C}P^3_{\mathfrak{m}_1 \oplus \mathfrak{m}_2} \times \mathrm{S}^2$	$\dfrac{\mathrm{Sp}(2)}{\mathrm{Sp}(1) \times \mathrm{U}(1)} \times \dfrac{\mathrm{SU}(2)}{\mathrm{U}(1)}$
8)	$\mathrm{S}^6_{\mathrm{irr}} \times \mathrm{S}^2$	$\dfrac{\mathrm{G}_2}{\mathrm{SU}(3)} \times \dfrac{\mathrm{SU}(2)}{\mathrm{U}(1)}$

Topological Remark: All these cosets are spin.

Geometrical Remark: Why one can focus only on non-symmetric cosets?

Step B – Topology

Proposition. Among the manifolds described in Table 1, only those appearing in the first four rows admit $\mathrm{Spin}(7)$-structures.

Examination of the topological restriction $p_1^2(M) - 4p_2(M) + 8\chi(M) = 0$

Proof.

- *Signature* of M^8: $\sigma(M^8) = \frac{1}{45}\langle 7p_2(M) - p_1^2(M), [M]\rangle$
- *$\hat{A}$-genus* of M^{4k} (spin) is the genus which has multiplicative sequences corresponding to the power series $F(t) = \dfrac{t/2}{\sinh(t/2)}$, $\quad \hat{A}(M^{4k}) := \Pi_{i=1}^{2k} \frac{x_i/2}{\sinh(x_i/2)} = 1 - \frac{1}{24}p_1 + \frac{1}{5760}(7p_1^2 - 4p_2) + \cdots$.
$$\hat{A}(M^8) = \frac{1}{5760}(7p_1^2(M^8) - 4p_2(M^8)).$$
- Dirac operator $D_+^g : \Gamma(\Sigma^+) \to \Gamma(\Sigma^-)$ and its index $\mathrm{ind}(D_+^g) = \dim\ker(D_+^g) - \dim\mathrm{coker}(D_+^g)$.
- Atiyah-Singer Index Theorem: $\mathrm{ind}(D_+^g) = \langle \hat{A}(M), [M]\rangle$. Moreover, if M (compact+spin) admits a metric of positive scalar curvature, then $\hat{A}(M) = 0$.

Step C – Invariant $\mathrm{Spin}(7)$-structures

- ▷ Given a coset $M^8 = G/H$ belonging in one of the first 4 rows of Table 1, we need to examine when the corresponding isotropy subalgebra $\mathfrak{h} \simeq \chi_*(\mathfrak{h})$ is a closed subalgebra of $\mathfrak{spin}(7)$
- ▷ Case by case examination (Case of $\mathrm{SU}(3)/\{e\}$ $\Rightarrow$ *M. Fernández* (1986))

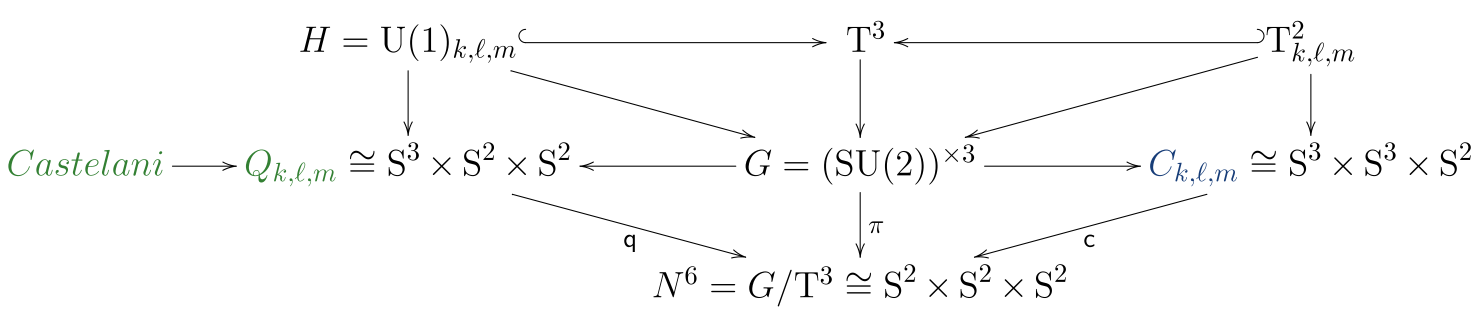
1. The homogeneous space $C_{k,\ell,m} = G/H = \dfrac{\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2)}{\mathrm{U}(1)^{k,\ell,m}}$

- We may assume $\gcd(k, \ell, m) = 1$ and $k, \ell, m \in \mathbb{Z}$ with $k \geqslant \ell \geqslant m \geqslant 0$ and $k > 0$.

  - Torus bundle over $\mathrm{S}^2 \times \mathrm{S}^2 \times \mathrm{S}^2$ (non-Kähler C-space)

$$\mathrm{T}^2_{k,\ell,m} \cong \mathrm{T}^3/H \longrightarrow C_{k,\ell,m} = G/H \xrightarrow{\mathsf{c}} N^6 = G/\mathrm{T}^3 = \mathrm{S}^2 \times \mathrm{S}^2 \times \mathrm{S}^2$$

  - $G := \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2), \quad H := \mathrm{U}(1)^{k,\ell,m} = \left\{(z^k, z^\ell, z^m) : z \in \mathrm{U}(1)\right\}$



The isotropy represeantation

$$\chi_*(\mathfrak{h}) = \mathrm{ad}(\mathfrak{h})|_{\mathfrak{m}} = \left\{ \begin{pmatrix} 0 & kx & & & & & & \\ -kx & 0 & & & & & & \\ & & 0 & \ell x & & & & \\ & & -\ell x & 0 & & & & \\ & & & & 0 & mx & & \\ & & & & -mx & 0 & & \\ & & & & & & 0 & 0 \\ & & & & & & 0 & 0 \end{pmatrix} : x \in \mathbb{R} \right\} \subset \mathfrak{so}(\mathfrak{m}) \simeq \mathfrak{so}(8)$$

A maximal torus of $\mathfrak{spin}(7)$

$$\mathfrak{t}^3 = \left\{ \begin{pmatrix} 0 & x & & & & & & \\ -x & 0 & & & & & & \\ & & 0 & y & & & & \\ & & -y & 0 & & & & \\ & & & & 0 & z & & \\ & & & & -z & 0 & & \\ & & & & & & 0 & -(x-y-z) \\ & & & & & & x-y-z & 0 \end{pmatrix} : x, y, z \in \mathbb{R} \right\} \subset \mathfrak{spin}(7).$$

Proposition. The space $C_{k,\ell,m} = G/H$, with $k \geqslant \ell \geqslant m \geqslant 0$, $k > 0$ and $\gcd(k, \ell, m) = 1$, admits G-invariant $\mathrm{Spin}(7)$-structures if and only if $k - \ell - m = 0$.

- For $k = \ell = 1$, $m = 0 \;\Rightarrow\; C_{1,1,0} \simeq \dfrac{\mathrm{SO}(4)}{\mathrm{SO}(2)} \times \mathrm{SU}(2) = \mathbb{V}_{4,2} \times \mathrm{S}^3 \simeq \mathrm{S}^3 \times \mathrm{S}^2 \times \mathrm{S}^3$

▷ Invariant $\mathrm{Spin}(7)$-structure? - Yes

- For $k = 1$, $\ell = m = 0 \;\Rightarrow\; C_{1,0,0} \simeq \mathrm{Spin}(4) \times \mathrm{SU}(2) = \dfrac{\mathrm{SU}(2) \times \mathrm{SU}(2)}{\{e\}} \times \dfrac{\mathrm{SU}(2)}{\mathrm{U}(1)} \simeq$ 6D Ledger-Obata space $\times\, \mathrm{S}^2 \simeq \mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^2$. $\left(\dfrac{\mathrm{SU}(2) \times \mathrm{SU}(2)}{\{e\}} \simeq \dfrac{\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2)}{\Delta\,\mathrm{SU}(2)}\right)$

▷ Invariant $\mathrm{Spin}(7)$-structure? - No

Theorem. The canonical presentations of compact, simply connected, almost effective homogeneous spaces admitting an invariant $\mathrm{Spin}(7)$-structure are exhausted by $\frac{\mathrm{SU}(3)}{\{e\}}$, the infinite family $C_{k,\ell,m}$, for $k = \ell + m$, and the Calabi-Eckmann manifold $\frac{\mathrm{SU}(3)}{\mathrm{SU}(2)} \times \mathrm{SU}(2)$.

Conclusion: There are just a few examples of compact simply connected homogeneous spaces admitting invariant $\mathrm{Spin}(7)$-structures.

Part B) On the type of a generic invariant $\mathrm{Spin}(7)$-structure

$M^8 = G/H = C_{k,\ell,m}$ admits a $\mathrm{Spin}(7)$-structure iff $k = \ell + m$

$\triangleright$ $C_{\ell+m,\ell,m}$, $\ell > m > 0$ $\triangleright$ $C_{\ell,\ell,0}$, $\ell > 0$, $m = 0$

- Assume that $k > \ell > m > 0$. Orthogonal decomposition $\mathfrak{m} = \mathfrak{m}_1 + \mathfrak{m}_2 + \mathfrak{m}_3 + \mathfrak{m}_4 + \mathfrak{m}_5$
- Q the normal metric on $C_{k,\ell,m}$ induced by the bi-invariant metric $\langle A, B\rangle = -2\mathrm{tr}(AB)$ on $\mathfrak{g}$.
- Up to scaling, any G-invariant metric on $C_{k,\ell,m}$ is given by

  $$(\, , \,)_{y_1,y_2,y_3,y_4,y_5,\theta} = y_1^2\, Q|_{\mathfrak{m}_1} + y_2^2\, Q|_{\mathfrak{m}_2} + y_3^2\, Q|_{\mathfrak{m}_3} + y_4^2\, Q|_{\mathfrak{m}_4^\theta} + y_5^2\, Q|_{\mathfrak{m}_5^\theta}, \quad y_1, \ldots, y_5 \in \mathbb{R},\ \theta \in [0, 2\pi]$$

- Space of G-invariant metrics on $C_{k,\ell,m}$, with $k > \ell > m > 0$, is 6-dimensional.

Invariant forms on $C_{\ell+m,\ell,m}$

- $\dim \Lambda^1(\mathfrak{m})^H = 2$, $\quad \dim \Lambda^2(\mathfrak{m})^H = 4$, $\quad \dim \Lambda^3(\mathfrak{m})^H = 8$, $\quad \dim \Lambda^4(\mathfrak{m})^H = 10$

Theorem. The 4-form

$$\begin{aligned}\Phi_{y_1,y_2,y_3,y_4,y_5} &= y_1^2 y_2^2 e^{1234} + y_1^2 y_3^2 e^{1256} - y_2^2 y_3^2 e^{3456} - y_1^2 y_4 y_5 e^{1278} + y_2^2 y_4 y_5 e^{3478} + y_3^2 y_4 y_5 e^{5678} \\ &\quad + y_1 y_2 y_3 y_4\, \beta_1 + y_1 y_2 y_3 y_5\, \beta_2,\end{aligned}$$

where

$$\beta_1 = e^{1357} - e^{1467} + e^{2457} + e^{2367}, \quad \beta_2 = e^{1368} + e^{1458} + e^{2468} - e^{2358},$$

defines an invariant $\mathrm{Spin}(7)$-structure on $C_{\ell+m,\ell,m}$, $\ell > m > 0$ of **mixed type**. It induces the normal metric $(\ ,\ )_{y_1,y_2,y_3,y_4,y_5}$ and its Lee form is given by

$$\begin{aligned}\vartheta_{y_1,y_2,y_3,y_4,y_5} &= \frac{2}{7\bar{c}_7}\Big(\ell m(y_2^2 y_3^2 - y_1^2 y_3^2 + 6 y_1^2 y_2^2) y_4^2 y_5^2 + m^2 (y_3^2 y_4^2 + y_1^2 y_4^2 + 4 y_1^2 y_3^2) y_2^2 y_5^2 \\ &\quad + 2\ell^2 (y_4^2 + 2 y_3^2) y_1^2 y_2^2 y_5^2\Big)\, e^8 - \frac{2(2l+m)}{7\bar{c}_8}\Big((2y_1^2 + y_5^2) y_2^2 y_3^2 y_4^2\Big)\, e^7.\end{aligned}$$

The characteristic connection

- Any any 8-dimensional manifold M endowed with a $\mathrm{Spin}(7)$-structure Φ admits a unique metric connection ∇ with totally skew-symmetric torsion T, satisfying $\nabla\Phi = 0$:

$$\nabla = \nabla^{h_\Phi} + \frac{1}{2}T, \quad T = -\delta\Phi - \frac{7}{6}\star(\vartheta \wedge \Phi).$$

Theorem. The invariant $\mathrm{Spin}(7)$-structure defined by the admissible 4-form $\Phi = \Phi_{1,\dots,1}$ on $C_{\ell+m,\ell,m}$, with $\ell > m > 0$, is of mixed type and its characteristic connection ∇ coincides with the canonical connection ∇^0 with respect to the naturally reductive structure induced by $g = (\ ,\ )_{1,\dots,1}$,

$$T^0(X,Y) = -[X,Y]_{\mathfrak{m}}, \quad \forall\ X,Y \in \mathfrak{m}.$$

In particular, its torsion form is parallel, i.e., $\nabla T = 0$.

The Calabi-Eckmann manifold $M^8 = G/H = \dfrac{\mathrm{SU}(3)}{\mathrm{SU}(2)} \times \mathrm{SU}(2)$

- Torus bundle over $\mathbb{C}P^2 \times \mathbb{C}P^1$ (non-Kähler C-space)
- $\mathrm{SU}(2) \subset \mathrm{G}_2 \subset \mathrm{Spin}(7) \;\Rightarrow$ Admits invariant $\mathrm{Spin}(7)$-structure ✓
- $\mathfrak{m} = \mathfrak{m}_1 + \mathfrak{m}_2 + \mathfrak{m}_3 = V^4 + \mathbb{R} + \mathbb{R}^3 = \mathfrak{n} + \mathbb{R}^3$,
- $\dim(\Lambda^1\mathfrak{m}^*)^H = 4, \quad \dim(\Lambda^2\mathfrak{m}^*)^H = 9, \quad \dim(\Lambda^3\mathfrak{m}^*)^H = 16, \quad \dim(\Lambda^4\mathfrak{m}^*)^H = 20.$
- Invariant metrics on M

  $g_{t_1,t_2,t_3,t_4,t_5} = t_1^2 \langle\cdot,\cdot\rangle|_{V^4} + t_2^2 \langle\cdot,\cdot\rangle|_{\mathbb{R}} + h_{t_3,t_4,t_5}, \quad h_{t_3,t_4,t_5} = t_3^2 (e^6 \otimes e^6) + t_4^2 (e^7 \otimes e^7) + t_5^2 (e^8 \otimes e^8).$

The type and the characteristic connection

Theorem. On the *Calabi-Eckmann manifold* $(\mathrm{SU}(3)/\,\mathrm{SU}(2))\times\mathrm{SU}(2)$, the invariant $\mathrm{Spin}(7)$-structure given by

$$\begin{aligned}\Phi_{t_1,t_2,t_3,t_4,t_5} \;=\;& t_1^4\, e^{1234} + t_1^2 t_2 t_3\,(e^{1256} - e^{3456}) + t_1^2 t_4 t_5 (e^{3478} - e^{1278}) + t_1^2 t_2 t_4 (e^{1357} + e^{2457}) \\ & + t_1^2 t_3 t_5 (e^{1368} + e^{2468}) + t_1^2 t_2 t_5 (e^{1458} - e^{2358}) + t_1^2 t_3 t_4 (e^{2367} - e^{1467}) \\ & + t_2 t_3 t_4 t_5 e^{5678},\end{aligned}$$

induces the general invariant metric $g_{t_1,\ldots,t_5}$, and it is of **mixed type** with Lee form

$$\vartheta_{t_1,t_2,t_3,t_4,t_5} = -\frac{1}{7}\left(\frac{2\,t_2\,(t_3^2+t_4^2+t_5^2)}{t_3 t_4 t_5}\,e^5 + \frac{4\sqrt{3}\,t_3\,(2t_1^2+t_2^2)}{t_1^2 t_2}\,e^6\right).$$

- Let Φ denote the invariant $\mathrm{Spin}(7)$-structure on $G/H = (\mathrm{SU}(3)/\,\mathrm{SU}(2))\times\mathrm{SU}(2)$ obtained by setting $t_i = 1$, for all $i = 1,\ldots,5$.

Theorem. $M^8 = G/H$ endowed with the metric induced by Φ is naturally reductive, and its canonical connection ∇^0 coincides with the characteristic connection ∇ of Φ. In particular, the torsion of ∇ is given by

$$T = -\sqrt{3}\,(e^{125} - e^{345}) + e^{678}$$

and it is ∇-parallel.

Thank You!

Compact homogeneous G_2-structures & weak G_2-structures

Invariant (non-weak) G_2-structures	Invariant weak G_2-structures
T^7	$W_{k,l} = \frac{SU_3}{S^1_{k,l}}$
$S^3 \times T^4$	$\mathbb{V}_{5,2} = \frac{SO_5}{SO_3^{st}}$
$S^3 \times S^3 \times S^1 = \frac{SU_2 \times SU_2 \times S^1}{\{e\}}$	$S^7 = \frac{Sp_2}{Sp_1} = \frac{Sp_2 \times U_1}{Sp_1 \times U_1} = \frac{Sp_2 \times Sp_1}{Sp_1 \times Sp_1} = \frac{Spin_7}{G_2}$
$S^3 \times S^3 \times S^1 = \frac{(SU_2 \times SU_2)}{\Delta SU_2} \times \frac{(SU_2 \times SU_2)}{\Delta SU_2} \times S^1$	$B^7 = \frac{SO_5}{SO_3^{irr}}$
$S^6 \times S^1$	$M_{k,l,m} := \frac{(S^3 \times S^3 \times S^3)}{(S^1 \times S^1)}$
$\mathbb{V}^{4,2} \times T^2$	$N_{k,l} = \frac{(SU_3 \times SU_2)}{(SU_2 \times S^1)}$
$S^5 \times T^2$	$W_{1,1} = \frac{SU_3 \times SU_2}{SU_2^c \times U_1}$
$\mathbb{F}_{1,2} \times S^1$	
$\mathbb{C}P^3 \times S^1$	