Regularity of two dimensional steady capillary gravity water waves

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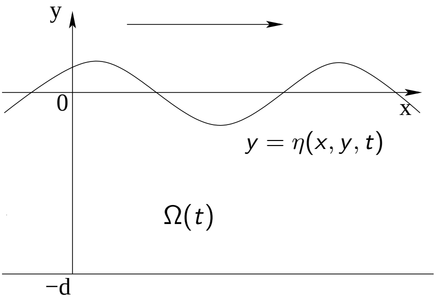
Model

- Incompressibility.
- No viscosity.
- The only external force acting on the bulk of the water is gravity.
- The free boundary is the graph of some function η.



Governing equations

Let $\Omega(t)$ be the domain occupied by the fluid at time t and $(u(x,y,t), v(x,y,t))$ be the velocity field.

Incompressibility implies

$$u_x + v_y = 0 \text{ in } \Omega(t).$$

Euler's equations:

$$\begin{aligned} u_t + uu_x + vu_y &= -P_x \\ v_t + uv_x + vv_y &= -P_y - g \end{aligned} \tag{1}$$

in $\Omega(t)$, where $P(x,y,t)$ denotes the pressure and g is the gravitation constant.

Boundary conditions

Kinematic boundary conditions:

- $v = \eta_t + u\eta_x$ on the free surface $S = \{(x, y); y = \eta(x, t)\}$.
- $v = 0$ on the bottom $B = \{y = -d\}$ for water waves of finite depth.
- $v \to 0$ uniformly as $y \to -\infty$ for water waves of infinite depth.

Dynamic condition (Young-Laplace equation):

$$P = P_0 - \sigma\kappa \quad \text{on} \quad S$$

where P_0 is the atmospheric pressure, $\sigma \geq 0$ is the surface tension coefficient and $\kappa = \left(\frac{\eta_x}{\sqrt{1+\eta_x^2}}\right)_x$ is the curvature of the free surface.

Traveling waves

We confine ourselves to traveling waves with constant speed $c > 0$. That is, the space-time dependence of the free surface, of the pressure, and of the velocity field has the form $(x - ct)$. We define the stream function $\psi(x, y)$ by

$$\psi_x = -v \quad \psi_y = u - c,$$

Let $\omega = v_x - u_y$ be the vorticity. Then $\Delta\psi = -\omega$. There is a function γ, called the vorticity function such that $\omega = \gamma(\psi)$. Thus

$$\Delta\psi = -\gamma(\psi)$$

Traveling waves

The kinematic boundary condition implies that ψ is constant on the free surface $y = \eta(x)$. We normalize ψ by choosing $\psi = 0$ on the free surface.

By Bernoulli's law, the quantity

$$E = \frac{(c-u)^2 + v^2}{2} + gy + P - \Gamma(\psi)$$

is a constant throughout the fluid, where $\Gamma(s) = \int_0^s \gamma(t)dt$ is the primitive function of γ.

On the free surface, $\Gamma(\psi) = 0$ and $P = P_0 - \sigma\kappa$. Therefore the dynamic boundary condition is equivalent to that

$$|\nabla\psi|^2 + 2gy - 2\sigma\kappa = Q$$

on the free surface, where Q is a constant.

Equivalent Formulation of the Problem

The flow may be described by the following free boundary problem:

$$\begin{aligned} &\Delta\psi = -\gamma(\psi) \quad \text{in} \quad \Omega \cap \{\psi > 0\}, \\ &|\nabla\psi|^2 + 2gy - 2\sigma\kappa = Q \quad \text{on} \quad S = \partial\{\psi > 0\}, \\ &\psi = 0 \quad \text{on} \quad S, \end{aligned} \tag{2}$$

where γ is the vorticity function, S is the graph of η, Q is a constant and $\kappa = \left(\frac{\eta_x}{\sqrt{1+\eta_x^2}}\right)_x$ is the curvature of S. By a translation, we may assume $Q = 0$.

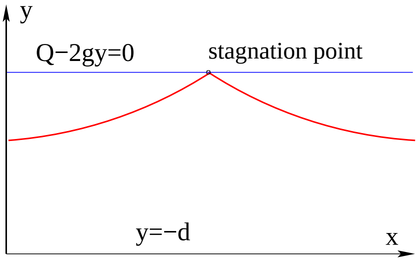
Extreme wave

Definition

Stagnation point: A point where $\nabla\psi = (0, 0)$.

Definition

Extreme wave: With a stagnation point on the free boundary.



Stokes conjecture

For irrotational flow without surface tension, i.e. $\gamma = 0$ and $\sigma = 0$.

Stokes conjecture(Stokes 1880): the profile of any extreme wave must have corners with included angle of 120 degrees at stagnation points.

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- Existence of extreme waves: Toland(1978), McLeod(1979).
- Stokes conjecture: Amick, Fraenkel and Toland(1982), Plotnikov(1982).
- Generalized Stokes conjecture Varvaruca and Weiss(2011).

Our problem

If the surface tension coefficient $\sigma > 0$, does singular free boundary point exist?

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Known results:

For irrotational water waves, i.e. $\gamma = 0$,

- B. Buffoni, E. N. Dancer, and J. F. Toland(2000) : $W^{2,2}$ free surfaces are analytic.
- Walter Craig and Ana-Maria Matei(2007): for 3-dimensional water waves, $C^{2,\alpha}$ free surfaces are analytic.

Our main result

Let ψ be a variational solution of equations:

$$\Delta\psi = -\gamma(\psi) \ \text{ in } \ \Omega \cap \{\psi > 0\},$$
$$|\nabla\psi|^2 + 2gy - 2\sigma\kappa = Q \ \text{ on } \ S = \partial\{\psi > 0\}.$$

If S is the graph of some $W^{1,1}$ function and the curvature is a Radon measure, then ψ is a classic solution and the free boundary $S = \partial\{\psi > 0\}$ is a $C^{2,\alpha}$ curve. Moreover, S is smooth if $\gamma \in C^{\infty}$, , S is analytic if γ is analytic.

Energy functional

Let Ω be a domain in $\mathbb{R}^2$, $\Gamma(t) = \int_0^t \gamma(s)ds$,
$\mathcal{V} = \{\psi \in W^{1,2}(\Omega), \psi \geq 0 \text{ and } \psi(x,y) = 0 \text{ if } y \geq 0\}$. For $\psi \in \mathcal{V}$ we define the Kinetic energy

$$K(\psi) = \int_\Omega \frac{1}{2}|\nabla\psi|^2 - \Gamma(\psi)dxdy,$$

the Potential energy

$$V(\psi) = \int_\Omega gy\chi_{\{\psi>0\}}dxdy + \sigma\mathcal{H}^1(\partial\{\psi>0\}),$$

and

$$\begin{aligned}\mathcal{L}(\psi) &= K(\psi) - V(\psi)\\ &= \int_\Omega \frac{1}{2}|\nabla\psi|^2 - \Gamma(\psi) - gy\chi_{\{\psi>0\}}dxdy - \sigma\mathcal{H}^1(\partial\{\psi>0\}).\end{aligned}$$

Domain variational formula

Let $\xi \in C_0^\infty(\Omega; \mathbb{R}^2)$, $\psi_\epsilon(\mathbf{x}) = \psi(\mathbf{x} + \epsilon\xi(\mathbf{x}))$. If ψ is a critical point of the energy function $\mathcal{L}$, i.e.

$$\delta\mathcal{L}(\psi)\xi = \lim_{\epsilon\to 0} \frac{1}{\epsilon}(\mathcal{L}(\psi_\epsilon) - \mathcal{L}(\psi)) = 0.$$

Euler-Lagrange equation

$$\int_\Omega (|\nabla\psi|^2 \mathrm{div}\xi - 2\nabla\psi D\xi\nabla\psi - 2\Gamma(\psi)\mathrm{div}\xi - 2gy\chi_{\{\psi>0\}}\mathrm{div}\xi - 2g\chi_{\{\psi>0\}}\xi_2) - 2\sigma\int_S \mathrm{div}_S\xi = 0.$$

Proposition

Any critical point ψ of the functional $\mathcal{L}$ is a classic solution to the two dimensional capillary water waves if its free boundary $\partial\{\psi > 0\}$ is a C^2 curve.

Domain variation solution

Definition

Let Ω be a domain in $\mathbb{R}^2$, $\Gamma(t) = \int_0^t \gamma(s)ds$. A function $0 \le \psi \in W^{1,2}_{\mathrm{loc}}(\Omega)$ is called a *domain variation solution of equation* (2), if $\psi \in C^0(\Omega) \cap C^2(\Omega \cap \{\psi > 0\})$ and

$$\int_\Omega (|\nabla\psi|^2 \mathrm{div}\xi - 2\nabla\psi D\xi\nabla\psi - 2\Gamma(\psi)\mathrm{div}\xi \\ - 2gy\chi_{\{\psi>0\}}\mathrm{div}\xi - 2g\chi_{\{\psi>0\}}\xi_2) - 2\sigma\int_S \mathrm{div}_S\xi = 0$$

for any $\xi(x) = (\xi_1(x), \xi_2(x)) \in C_0^\infty(\Omega; \mathbb{R}^2)$.

Assumptions

We assume that $\gamma \in L^\infty$, the free boundary is the graph of some function $\eta \in W^{1,1}_{\mathrm{loc}}$ and that the curvature $\kappa = \left(\frac{\eta_x}{\sqrt{1+(\eta_x)^2}}\right)_x$ is a Radon measure on the free surface. We also assume that S can be touched at every point from below by a ball of fixed radius $\kappa_0^{-1} > 0$, which implies that $\kappa \geq -\kappa_0$ in the sense of measures. This assumption is justified by the expected free boundary condition

$$|\nabla\psi|^2 + 2gy - 2\sigma\kappa = Q \quad \text{on} \quad S = \partial\{\psi > 0\}.$$

Remarks on the assumptions

Remark

In some cases we can prove that for a domain variation solution that $\kappa > -\kappa_0$ *in the sense of measure.*
For example, if $\psi_y < 0$*, i.e.* $u < c$ *in* $\{\psi > 0\}$*, then* κ *is bounded from below.*

Remark

Assuming that a general simple curve S *can be touched at every point by a ball of fixed radius* $\kappa_0^{-1} > 0$ *contained in the water phase, all our results extend to the simple curve case too.*

A Bonnet type monotonicity formula

Let ψ be a domain variation solution and
$\Phi_{\mathbf{x}}(r) = r^{-1} \int_{B_r(\mathbf{x})} |\nabla\psi|^2 d\mathbf{x}$ for $\mathbf{x} \in S$. Then $\Phi' \geq -C_3 r^{1/2} \Phi^{1/2}$.
Thus

Proposition

The limit $\lim_{r \to 0+} \Phi_x(r) \in [0, +\infty)$ *exists for every* $\mathbf{x} \in \partial\{\psi > 0\}$.

Proposition

$\Phi_{\mathbf{x}}(0+)$ *is upper semi-continuous with respect to* $\mathbf{x}$.

Scaling

Let $\mathbf{x}^0 \in S$, $\psi_r(\mathbf{x}) = r^{-1/2}\psi(\mathbf{x}_0 + r\mathbf{x})$. Then $S_r = \{\mathbf{x} : \mathbf{x}_0 + r\mathbf{x} \in S\}$ is the graph of $\eta_r(\mathbf{x}) = \frac{\eta(\mathbf{x}_0 + r\mathbf{x}) - \eta(\mathbf{x}_0)}{r}$ and $\kappa_r(\mathbf{x}) = r\kappa(\mathbf{x}_0 + r\mathbf{x}) \geq -r\kappa_0$. ψ_r is a domain variation solution in the sense

$$
\begin{aligned}
&\int_{\Omega_r} (|\nabla\psi_r|^2 \mathrm{div}\xi - 2\nabla\psi_r D\xi\nabla\psi_r - 2r\Gamma(r^{1/2}\psi_r)\mathrm{div}\xi \\
&- 2g(y_0 + ry)r\chi_{\{\psi_r>0\}}\mathrm{div}\xi - 2g(y_0 + ry)r\chi_{\{\psi_r>0\}}\xi_2) \qquad (3) \\
&- 2\sigma \int_{S_r} \mathrm{div}_{S_r}\xi = 0,
\end{aligned}
$$

where $\Omega_r = \{\mathbf{x} : \mathbf{x}^0 + r\mathbf{x} \in \Omega\}$.

Convergence of ψ_r

Proposition

There exists a subsequence $\{\psi_{r_k}\}$ *such that* $\psi_{r_k} \to \psi_0$ *weakly in* $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2)$ *and strongly in* $L^2_{\mathrm{loc}}(\mathbb{R}^2)$.

Proposition

There exists a $C_R < \infty$ *such that* $||\psi_r||_{C^{1/2}(B_R(0))} \leq C_R$*; here* C_R *is independent of* r.

Proposition

$\psi_{r_k} \to \psi_0$ *strongly in* $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2)$.

Convergence of the curvature term

Lemma

Let $\nu = (\cos\theta(x), \sin\theta(x))$ *be the unit outward normal vector of* $\{\psi > 0\}$ *at point* $\mathbf{x} = (x, \eta(x))$ *for a.e.* $\mathbf{x}$. *Then for each* x_0, $\mathrm{esslim}_{x\to x_0+}\theta(x)$ *and* $\mathrm{esslim}_{x\to x_0-}\theta(x)$ *exist.*

Proposition

$\lim_{k\to\infty} \int_{S_{r_k}} \mathrm{div}_{S_{r_k}}\xi = (T(0-) - T(0+)) \cdot \xi(0)$ *for* $\xi \in C_0^\infty(\Omega;\ \mathbb{R}^2)$, *where* $T(0+)$ *and* $T(0-)$ *denote the right and left unit tangent vector of* S *at* 0, *respectively.*

Characterization of blow-up limits

Recall that

$$\int_{\Omega_r} (|\nabla \psi_r|^2 \mathrm{div}\xi - 2\nabla\psi_r D\xi \nabla\psi_r - 2r\Gamma(r^{1/2}\psi_r)\mathrm{div}\xi$$
$$- 2g(y_0 + ry)r\chi_{\{\psi_r>0\}}\mathrm{div}\xi - 2g(y_0 + ry)r\chi_{\{\psi_r>0\}}\xi_2) \qquad (4)$$
$$- 2\sigma \int_{S_r} \mathrm{div}_{S_r}\xi = 0.$$

Letting $r \to 0$, we obtain that ψ_0 is a domain variation solution in the sense

$$\int (|\nabla\psi_0|^2 \mathrm{div}\xi - 2\nabla\psi_0 D\xi \nabla\psi_0) + 2\sigma\xi(0)\cdot(T(0+) - T(0-)) = 0.$$

Cusp points

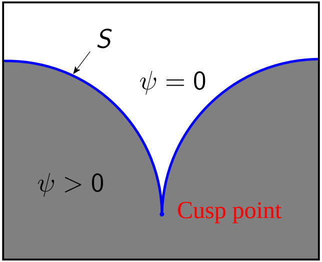
Proposition

For each free boundary point **x***, the blow-up limit* $\psi_0(r,\theta) = cr^{1/2}\cos(\frac{\theta}{2} + \frac{3\pi}{4})$*,* $\theta \in (-\frac{5\pi}{2}, -\frac{\pi}{2}]$*,* $c = 2\sqrt{\frac{2\sigma}{\pi}}$ *or* $c = 0$*.*

Definition

Let $\mathbf{x} \in S$. We call **x** a cusp point if the blow-up limit at **x** is $\psi_0(r,\theta) = 2\sqrt{\frac{2\sigma}{\pi}} r^{1/2}\cos(\frac{\theta}{2} + \frac{3\pi}{4})$. We denote by $\mathcal{C}$ the set of cusp points.

Cusps are isolated



$\Phi_{\mathbf{x}}(0+) = C_0 > 0$ for $\mathbf{x} \in \mathcal{C}$ and $\Phi_{\mathbf{x}}(0+) = 0$ for $\mathbf{x} \notin \mathcal{C}$. From the upper semi-continuity of $\Phi_{\mathbf{x}}(0+)$ with respect to $\mathbf{x}$ we obtain that $\mathcal{C}$ is closed. Moreover,

Proposition

Cusp points are isolated.

Regularity outside the set $\mathcal{C}$

Let $U \Subset \Omega$ be a domain such that $\overline{U} \cap \mathcal{C} = \emptyset$.

Proposition

$S \cap U$ is a C^1curve.

Let $0 < \alpha < 1$. We define $\Phi_{\mathbf{x}}^{\alpha+1}(r) = r^{-(\alpha+1)} \int_{B_r(\mathbf{x})} |\nabla \psi|^2$. There exist $r_0 > 0$ and $C < \infty$ such that $\Phi_{\mathbf{x}}^{\alpha+1}(r) \leq C$ for $\mathbf{x} \in S \cap U$ and $r \leq r_0$.

In the proof of $\Phi_{\mathbf{x}}^{1}(r) < C$, we used the Wirtinger's inequality

$$\int_0^{2\pi} \left| \frac{\partial}{\partial \theta} \psi(r,\theta) \right| d\theta \geq \frac{1}{4} \int_0^{2\pi} \psi^2(r,\theta) d\theta.$$

By the C^1 regularity, there exist an $r_0 > 0$ and a cone K with opening angle $\frac{2\alpha\pi}{\alpha+1}$ and vertex at $\mathbf{x}$ such that $K \cap B_{r_0}(\mathbf{x}) \subset \{\psi = 0\}$. So we have

$$\int_0^{2\pi} \left| \frac{\partial}{\partial \theta} \psi(r,\theta) \right| d\theta \geq \frac{(1+\alpha)^2}{4} \int_0^{2\pi} \psi^2(r,\theta) d\theta.$$

Proposition

$S \cap U$ is a $C^{1,\alpha}$ curve.

Theorem

$S \cap U$ is a $C^{2,\alpha}$ curve. Moreover if $\gamma \in C^\infty$, then $S \cap U$ is smooth and $\psi \in C^\infty(\overline{\{\psi > 0\}} \cap U)$.

Cusp does not exist

At a cusp point, the blow-up limit is

$$\psi_0(r,\theta) = 2\sqrt{\frac{2\sigma}{\pi}}r^{1/2}\cos(\frac{\theta}{2}+\frac{3\pi}{4}).$$

Gradient estimate near cusp point:

Lemma

Let $\mathbf{x}^0$ be a cusp point. There exist constants $\tau > 0$ and $\delta > 0$ such that $|\nabla\psi(\mathbf{z})| \geq \frac{\tau}{|\mathbf{z}-\mathbf{x}^0|^{1/2}}$ for all $\mathbf{z} \in S$ and $|\mathbf{z}-\mathbf{x}^0| < \delta$.

Using the boundary condition $|\nabla\psi(\mathbf{x})|^2 + 2gy - 2\sigma\kappa = constant$ on $S \setminus \mathcal{C}$ we obtain

Proposition

$\mathcal{C} = \emptyset$.

Our main theorem

Theorem

Each domain variation solution ψ is a classic solution to the two-dimensional steady capillary gravity water waves problem and the free surface $S = \partial\{\psi > 0\}$ is a $C^{2,\alpha}$ curve. Moreover, S is smooth if $\gamma \in C^\infty$, S is analytic if γ is analytic.

Summary

- At each free boundary point **x**, the blow-up limit of ψ is $\psi_0(r,\theta) = 2\sqrt{\frac{2\sigma}{\pi}} r^{1/2} \cos(\frac{\theta}{2} + \frac{3\pi}{4})$ or $\psi_0 \equiv 0$.
- If $\psi_0(r,\theta) = 2\sqrt{\frac{2\sigma}{\pi}} r^{1/2} \cos(\frac{\theta}{2} + \frac{3\pi}{4})$, **x** is called a cusp point. Cusp points are isolated.
- $S \setminus \mathcal{C}$ is smooth, where $\mathcal{C}$ denotes the set of cusp points.
- Cusp points do not exist.