Steady water waves with compactly supported vorticity

Samuel Walsh[†]
Jalal Shatah[†] Chongchun Zeng[‡]

[†]Courant Institute, New York University
[‡]Georgia Institute of Technology

Nonlinear waves and interface problems
Centre for Mathematical Sciences, Lund University

Introduction

The study of steady water waves began with investigations of the irrotational case, i.e. where the velocity field has gradient structure. This is justified on physical grounds (by Kelvin's circulation theorem), but equally important, it is extremely convenient mathematically. This is because the interior dynamics are captured by Laplace's equation, and thus one can push the entire problem to the boundary, where it typically becomes nonlocal.

Irrotational waves still comprise the vast majority of current research into water waves; the literature for them is impossibly voluminous. (Yet many fundamental problems remain.)

On the other hand, rotational steady waves occur frequently in nature (due to wind forcing, heterogeneous density, etc.) Here, significant progress has only been made recently, particularly following the breakthrough work of Constantin and Strauss in 2004.

We now enjoy a bounty of existence results for various regimes of rotational waves, many of them proved by people in the audience today. Clearly, though, the rotational theory is far less explored than the irrotational.

One common feature of all of the existence results for rotational waves is that the vorticity is not compactly supported.

For example, it is easy to see why this is the case for any work following the Constantin–Strauss program. Briefly, they assume an absence of stagnation which

(i) allows you to take the streamlines as a vertical coordinate to fix the domain (the Dubreil-Jacotin transform), and
(ii) implies the vorticity ω is a function of the stream function ψ_{rel},

$$\omega = \gamma(\psi_{\mathrm{rel}}).$$

Together, these facts immediately force $\operatorname{supp}\omega$ to be non-compact unless ω vanishes identically.

One common feature of all of the existence results for rotational waves is that the vorticity is not compactly supported.

For example, it is easy to see why this is the case for any work following the Constantin–Strauss program. Briefly, they assume an absence of stagnation which

(i) allows you to take the streamlines as a vertical coordinate to fix the domain (the Dubreil-Jacotin transform), and

(ii) implies the vorticity ω is a function of the stream function ψ_{rel},

$$\omega = \gamma(\psi_{\text{rel}}).$$

Together, these facts immediately force $\operatorname{supp}\omega$ to be non-compact unless ω vanishes identically.

One common feature of all of the existence results for rotational waves is that the vorticity is not compactly supported.

For example, it is easy to see why this is the case for any work following the Constantin–Strauss program. Briefly, they assume an absence of stagnation which

(i) allows you to take the streamlines as a vertical coordinate to fix the domain (the Dubreil-Jacotin transform), and

(ii) implies the vorticity ω is a function of the stream function ψ_{rel},

$$\omega = \gamma(\psi_{\mathrm{rel}}).$$

Together, these facts immediately force $\operatorname{supp}\omega$ to be non-compact unless ω vanishes identically.

One common feature of all of the existence results for rotational waves is that the vorticity is not compactly supported.

For example, it is easy to see why this is the case for any work following the Constantin–Strauss program. Briefly, they assume an absence of stagnation which

(i) allows you to take the streamlines as a vertical coordinate to fix the domain (the Dubreil-Jacotin transform), and

(ii) implies the vorticity ω is a function of the stream function ψ_{rel},

$$\omega = \gamma(\psi_{\mathrm{rel}}).$$

Together, these facts immediately force $\operatorname{supp}\omega$ to be non-compact unless ω vanishes identically.

As far as we are aware, even those works that do not rely on the Dubreil-Jacotin transform, also have this feature, particularly those that construct their solutions as perturbations of shear flows, i.e. flows where the streamlines (including the free surface) are flat.

Moreover, the relation $\omega = \gamma(\psi_{\mathrm{rel}})$ is either imposed by fiat (as in the critical layer work of Wahlén, Ehrnström, and Villari), or arises some other way (e.g., as a Lagrange multiplier in the variational formulation of Burton, Toland, and Buffoni.)

The combination of a near shear flow, and a functional dependence of ω on ψ_{rel} makes it very hard to construct compactly supported vorticity, since you must arrange that all streamlines on which $\omega \neq 0$ close up upon bifurcation — this is not easy.

As far as we are aware, even those works that do not rely on the Dubreil-Jacotin transform, also have this feature, particularly those that construct their solutions as perturbations of shear flows, i.e. flows where the streamlines (including the free surface) are flat.

Moreover, the relation $\omega = \gamma(\psi_{\mathrm{rel}})$ is either imposed by fiat (as in the critical layer work of Wahlén, Ehrnström, and Villari), or arises some other way (e.g., as a Lagrange multiplier in the variational formulation of Burton, Toland, and Buffoni.)

The combination of a near shear flow, and a functional dependence of ω on ψ_{rel} makes it very hard to construct compactly supported vorticity, since you must arrange that all streamlines on which $\omega \neq 0$ close up upon bifurcation — this is not easy.

In summary, there is a vast body of work on the irrotational case ($\omega \equiv 0$); a rapidly growing body of work for the rotational case (where ω does not even vanish at infinity!); and a no man's land in-between:

irrotational $\omega \equiv 0$	$\longleftarrow$	(????)	$\longrightarrow$	rotational $\operatorname{supp} \omega$ unbounded, ω does not decay at ∞.

The subject of the present talk is an existence theory for two categories of steady waves whose vorticity has compact support, developed jointly with Jalal Shatah and Chongchun Zeng.

In summary, there is a vast body of work on the irrotational case ($\omega \equiv 0$); a rapidly growing body of work for the rotational case (where ω does not even vanish at infinity!); and a no man's land in-between:

irrotational $\omega \equiv 0$	$\longleftarrow$	(????)	$\longrightarrow$	rotational $\operatorname{supp} \omega$ unbounded, ω does not decay at ∞.

The subject of the present talk is an existence theory for two categories of steady waves whose vorticity has compact support, developed jointly with Jalal Shatah and Chongchun Zeng.

Formulation

Let's now step back and state things more carefully.

Consider a two-dimensional traveling wave moving with speed $c > 0$ in the x_1-direction and with infinite depth in the negative x_2-direction. Adopting a reference frame moving with the wave eliminates any time dependence, so we may suppose that the fluid occupies a domain

$$\Omega := \{(x_1, x_2) \in \mathbb{R}^2 : x_2 < 1 + \eta(x_1)\},$$

for some function η.

The vorticity ω of a velocity field $\mathbf{v} \in L^2(\Omega;\mathbb{R}^2)$ is the distribution

$$\omega := \partial_{x_1} v_2 - \partial_{x_2} v_1.$$

When ω has compact support, we may define the vortex strength to be

$$\epsilon := \int_\Omega \omega \, dx.$$

For ω a finite measure, we this read this as $\epsilon = \omega(\Omega)$.

If **v** is divergence free (incompressible) and the vorticity has compact support, then in any simply connected region lying outside the support of ω, we have the decomposition

$$\mathbf{v} = \nabla \varphi_{\mathscr{H}} + \epsilon \nabla \Theta,$$

where $\varphi_{\mathscr{H}}$ is a harmonic function and

$$\nabla \Theta = \frac{1}{2\pi} \nabla^{\perp} \left(\log|\cdot| * \frac{\omega}{\epsilon} \right) - \frac{1}{2\pi} \nabla^{\perp} \log|\cdot - 2\mathbf{e}_2|.$$

The purpose of the second logarithm term is to ensure that $\Theta \in \dot{H}^1$. Physically, it corresponds to the standard trick of introducing a phantom point vortex in the air region with equal but opposite vortex strength to correct for the lack of integrability of the fundamental solution of Laplace's equation in $\mathbb{R}^2$.

In order for $\mathbf{v}$ to be the velocity for a traveling water wave, it must be a (weak) solution of the steady incompressible Euler's equations in the fluid domain,

$$-c\partial_{x_1}\mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v} = -\nabla p + g\mathbf{e}_2,$$

where p is the pressure and g is the gravitational constant. When ω has some regularity, we may take the curl of this equation to get the steady vorticity equation

$$-c\partial_{x_1}\omega + \nabla\cdot(\omega\mathbf{v}) = 0,$$

which expresses the fact that the vorticity is transported by the flow.

Main idea: If ω is supported away from the free surface, then we can formulate the boundary motion nearly as we would for an irrotational flow (using the decomposition.) The interior dynamics are determined by the vorticity equation. Finally, we couple the two to produce a closed system.

For instance, recall that there are two conditions on the boundary: the dynamic condition, and the kinematic condition. The first of these (along with Bernoulli's theorem) states that if $\mathbf{v} = \nabla\Phi$, then

$$-c\partial_{x_1}\Phi + \frac{1}{2}|\nabla\Phi|^2 + gx_2 + \alpha^2\kappa = b \qquad \text{on } \{x_2 = 1 + \eta(x_1)\},$$

where b is a constant, $\alpha^2 > 0$ is the coefficient of surface tension, and κ is the mean curvature of the surface.

The kinematic condition is the requirement that the normal velocity of the boundary agree with that of the fluid. Writing this in terms of Φ yields

$$0 = c\eta' + (1, \eta') \cdot \nabla\Phi \qquad \text{on } \{x_2 = 1 + \eta(x_1)\}.$$

For instance, recall that there are two conditions on the boundary: the dynamic condition, and the kinematic condition. The first of these (along with Bernoulli's theorem) states that if $\mathbf{v} = \nabla\Phi$, then

$$-c\partial_{x_1}\Phi + \frac{1}{2}|\nabla\Phi|^2 + gx_2 + \alpha^2\kappa = b \qquad \text{on } \{x_2 = 1 + \eta(x_1)\},$$

where b is a constant, $\alpha^2 > 0$ is the coefficient of surface tension, and κ is the mean curvature of the surface.

The kinematic condition is the requirement that the normal velocity of the boundary agree with that of the fluid. Writing this in terms of Φ yields

$$0 = c\eta' + (1, \eta') \cdot \nabla\Phi \qquad \text{on } \{x_2 = 1 + \eta(x_1)\}.$$

If the support of the vorticity is a positive distance from the boundary, we may use the decomposition to take $\Phi = \varphi_{\mathscr{H}} + \Theta$.

Let φ be the trace of $\varphi_{\mathscr{H}}$ on the free surface, and $\nabla\vartheta$ be the trace of $\nabla\Theta$, i.e.,

$$\varphi(x_1) = \varphi_{\mathscr{H}}(x_1, \eta(x_1)), \qquad (\nabla\vartheta)(x_1) := (\nabla\Theta)(x_1, \eta(x_1)).$$

Note: hereafter, we write $\mathscr{H}(\eta)f$, or $\mathscr{H}f$, or $f_{\mathscr{H}}$, to denote the harmonic extension into Ω of a function $f = f(x_1)$ defined on the surface.

The familiar Zakharov Hamiltonian formulation of the dynamic boundary condition then becomes

$$
\begin{aligned}
0 = &-c\left(\varphi' + \epsilon(1,\eta')\cdot\nabla\vartheta\right) + \frac{1}{2}\left(\varphi' + \epsilon(1,\eta')\cdot\nabla\vartheta\right)^2 \\
&- \frac{1}{2(1+(\eta')^2)}\left(\mathscr{G}(\eta)\varphi + \eta'\varphi' + \epsilon(1+(\eta')^2)\partial_{x_1}\vartheta\right)^2 \\
&+ g(\eta+1) + \alpha^2\kappa(\eta) - b.
\end{aligned} \tag{1}
$$

Here

$$
\frac{1}{\sqrt{1+(\eta')^2}}\mathscr{G}(\eta) := \mathscr{N}(\eta),
$$

and $\mathscr{N}(\eta)$ is the Dirichlet-to-Neumann operator on Ω. Note that this is a nonlocal problem posed on $\mathbb{R}$, or S^1 for the periodic case.

Likewise, the kinematic equation boils down to

$$0 = c\eta' + \mathscr{G}(\eta)\varphi + \epsilon(1,\eta') \cdot \nabla\vartheta. \tag{2}$$

Together (1) and (2) describe the motion of the boundary. Note that taking $\epsilon = 0$ reduces them to the irrotational case. We shall therefore treat ϵ as our parameter of bifurcation.

Vortex patches

Today, I will talk about one of the classes of vorticity that we studied, namely the vortex patch. These are solutions where $\omega \in L^2(\Omega) \cap L^1(\Omega)$ and is supported in a compact region $D \subset\subset \Omega$. We will require that ω be smooth in D but not necessarily over ∂D.

Reformulating the vorticity equation in terms of the relative stream function, what results is a nonlinear elliptic free boundary problem, very much in the same vein as Constantin–Strauss. However, rather than perturbing from a shear flow, our point of bifurcation will be radial solutions on a ball. Moreover, we use conformal mappings to fix the boundary of the patch rather than streamline coordinates.

To couple the vorticity equation to the boundary motion requires that some matching be done on ∂D.

Vortex patches

Today, I will talk about one of the classes of vorticity that we studied, namely the vortex patch. These are solutions where $\omega \in L^2(\Omega) \cap L^1(\Omega)$ and is supported in a compact region $D \subset\subset \Omega$. We will require that ω be smooth in D but not necessarily over ∂D.

Reformulating the vorticity equation in terms of the relative stream function, what results is a nonlinear elliptic free boundary problem, very much in the same vein as Constantin–Strauss. However, rather than perturbing from a shear flow, our point of bifurcation will be radial solutions on a ball. Moreover, we use conformal mappings to fix the boundary of the patch rather than streamline coordinates.

To couple the vorticity equation to the boundary motion requires that some matching be done on ∂D.

Vortex patches

Today, I will talk about one of the classes of vorticity that we studied, namely the vortex patch. These are solutions where $\omega \in L^2(\Omega) \cap L^1(\Omega)$ and is supported in a compact region $D \subset\subset \Omega$. We will require that ω be smooth in D but not necessarily over ∂D.

Reformulating the vorticity equation in terms of the relative stream function, what results is a nonlinear elliptic free boundary problem, very much in the same vein as Constantin–Strauss. However, rather than perturbing from a shear flow, our point of bifurcation will be radial solutions on a ball. Moreover, we use conformal mappings to fix the boundary of the patch rather than streamline coordinates.

To couple the vorticity equation to the boundary motion requires that some matching be done on ∂D.

Using a (local) bifurcation theory argument, we are able to construct a curve of small-amplitude, small vorticity, slow speed, and small patch solutions in the neighborhood of the trivial solution.

The main theorem in this direction is the following.

Theorem (Vortex patch local bifurcation)

For $s > 3/2$, there exists a C^1 surface of solutions to the vortex patch problem

$$\mathscr{S}_{\mathrm{loc}} = \{(\eta(\epsilon,\delta,\tau), \varphi(\epsilon,\delta,\tau), c(\epsilon,\delta,\tau), \omega(\epsilon,\delta,\tau)) : (\epsilon,\delta,\tau) \in \mathscr{U}\},$$

where $\mathscr{U} := [0,\epsilon_0) \times [0,\delta_0) \times [0,\tau_0)$ *for some* $\epsilon_0, \delta_0, \tau_0 > 0$,

$$\mathscr{S}_{\mathrm{loc}} \subset H^{s+1}_{\mathrm{e}}(\mathbb{R}) \times (\dot{H}^{s}_{\mathrm{e}}(\mathbb{R}) \cap \dot{H}^{1/2}_{\mathrm{e}}(\mathbb{R})) \times \mathbb{R} \times H^{s-1/2}_{\mathrm{e}}(D).$$

The parameterization is such that

$$(\eta(0,0,0), \varphi(0,0,0), c(0,0,0), \omega(0,0,0)) = (0,0,0,0),$$

and the support of the vorticity $\omega(\epsilon,\delta,s)$ *is an* H^s *domain* $D(\epsilon,\delta,s)$ *with asymptotic form:*

$$\partial D(\epsilon,\delta,s) = \{\delta(\cos\theta + s\sin(2\theta), \sin\theta - s\cos(2\theta)) + o(\epsilon,\delta,s) : \theta \in [0,2\pi]\}.$$

Some remarks.

- The evenness assumption will turn out to be crucial, for it allows us to infer the finite dimensionality of the linearized elliptic problem in the patch. Recall, though, that even symmetry is often expected for traveling waves.
- A global bifurcation theorem seems to be very hard. Note that we would need to prove compactness properties of the linearized operator not only at 0, but anywhere along the continuum. We shall see that this is very tricky for the elliptic PDE for the vortex dynamics.

Some remarks.

- The evenness assumption will turn out to be crucial, for it allows us to infer the finite dimensionality of the linearized elliptic problem in the patch. Recall, though, that even symmetry is often expected for traveling waves.
- A global bifurcation theorem seems to be very hard. Note that we would need to prove compactness properties of the linearized operator not only at 0, but anywhere along the continuum. We shall see that this is very tricky for the elliptic PDE for the vortex dynamics.

For the remainder of the talk, I will precisely formulate and outline the proof of the local bifurcation theorem for the vortex patch; the point vortex proof (at least the local one) is much simpler, and can be seen as a limiting case.
Recall that the total vorticity in this regime is

$$\epsilon := \int_\Omega \omega \, dx.$$

We construct waves as perturbations of the trivial solution where the water is totally stationary and the free surface entirely flat; accordingly, $\epsilon \ll 1$.

The patch $D := \operatorname{supp}(\omega)$ will be a perturbation of B_δ, the ball of radius δ centered at the origin, where $0 < \delta \ll 1$.

It will prove useful to impose the ansatz

$$\omega(x) = \frac{\epsilon}{\delta^2}\widetilde{\omega}(\frac{x}{\delta}),$$

where

$$\int_\Omega \widetilde{\omega}\, dx = 1.$$

Notice that if ω is supported on B_δ, then $\widetilde{\omega}$ is supported on B_1.

Notation. Here and afterwards, we will designate with tildes quantities that have been scaled in ϵ and δ.

Stream function problem

The incompressibility of $\mathbf{v}$ permits us to introduce a stream function ψ defined up to a constant by the requirement that

$$\nabla^{\perp}\psi = \mathbf{v}.$$

It follows immediately from the definition that $\Delta\psi = \omega$, and so inverting the Laplacian we see that ψ may be decomposed as

$$\psi = h - \frac{1}{2\pi}\log|\cdot| * \omega + \frac{1}{2\pi}\epsilon\log|\cdot - 2\mathbf{e}_2|,$$

where $h \in \dot{H}^1(\Omega)$ is a harmonic conjugate of $\varphi_{\mathscr{H}}$

Recall that the vorticity dynamics of a two-dimensional flow is governed by the equation

$$\omega_t + \mathbf{v} \cdot \nabla\omega = 0.$$

Assuming that ω has the traveling wave ansatz with speed c, we find that

$$(\mathbf{v} - c\mathbf{e}_1) \cdot \nabla\omega = 0,$$

which is equivalent to the statement that

$$\nabla^\perp f \cdot \nabla\omega = 0,$$

where $f := \psi + cx_2$ is the relative stream function.

Before proceeding further, it will be convenient to scale the physical variables and the relative stream function as we have the vorticity. With that in mind, define $D_0 := \frac{1}{\delta} D$ and $\widetilde{f} : D_0 \to \mathbb{R}$ by

$$f(x) =: \epsilon \widetilde{f}(\frac{x}{\delta}).$$

Note that, as $\Delta f = \omega$,

$$\Delta \widetilde{f} = \widetilde{\omega} \text{ in } D_0, \qquad \int_{D_0} \Delta \tilde{f}\, dx = 1.$$

One way to ensure the above equation is satisfied is to require that the vortex lines and streamline coincide, i.e.

$$\Delta \widetilde{f} = \widetilde{\omega} = \gamma(\widetilde{f}) \qquad \text{a.e. in } D_0,$$

for a some function γ called the vorticity strength function; this is the standard approach for rotational waves.

As $D_0 = \operatorname{supp}(\widetilde{\omega})$, ∂D is a vortex line ($\widetilde{\omega}|_{\partial D_0} = 0$) and thus a streamline. Without loss we may assume that $f|_{\partial D} = 0$ so that, in total, the scaled relative stream function $\widetilde{f} \in \dot{H}^1_0(D_0)$ is the solution to the elliptic PDE

$$\begin{cases} \Delta \widetilde{f} = \gamma(\widetilde{f}) & \text{in } D_0 \\ \widetilde{f} = 0 & \text{on } \partial D_0. \end{cases}$$

To be able to infer the existence of solutions to the problem above in a neighborhood of the trivial flow, we must impose some conditions on γ. In particular, we need to guarantee that the linearized operator is non-degenerate. Thus we assume

$$\gamma \in C^2(\mathbb{R}), \qquad \gamma(0) = 0, \qquad \gamma'(0) < 0, \qquad \gamma > 0 \text{ on } \mathbb{R}^-,$$

$$\text{there exists a negative radial solution } \widetilde{f}^* \text{ with } D_0 = B_1,$$

$$\Delta - \gamma'(\widetilde{f}^*) \text{ is non-degenerate.}$$

These are very mild, particularly the third requirement.

We can now find the vorticity given D_0, but how do we determine D_0? Let Γ be a conformal mapping with domain the unit ball $B_1 \subset \mathbb{C}$, and satisfying

$$\partial_{\bar{z}}\Gamma = 0 \text{ in } B_1, \qquad \Gamma(0) = 0, \qquad \Gamma'(0) = 1.$$

By identifying $z = x_1 + ix_2 \in \mathbb{C}$ with the point $(x_1, x_2) \in \mathbb{R}^2$, we may view the dilated domain D_0 as the image of B_1 under Γ, $D_0 := \Gamma(B_1)$, and the unscaled domain $D := \delta D_0$.

As we are interested in vortex patches that are perturbations of B_δ, we think of $|\Gamma(z) - z|$ as being small throughout D_0.

The first statement is just the conformality, while the second fixes the origin. The third is made in order to eliminate a certain redundancy: For each $\sigma > 0$, (δ, Γ) and $(\delta/\sigma, \sigma\Gamma)$ result in the same patch D. By fixing $\Gamma'(0) = 1$, we exclude all but $\sigma = 1$.

We also impose some symmetry properties for D. Specifically, we require that D is symmetric over the x_1-axis, which, in terms of Γ, is equivalent to

$$\operatorname{Re}\Gamma \text{ is odd in } x_1, \qquad \text{and} \qquad \operatorname{Im}\Gamma \text{ is even in } x_1.$$

The first statement is just the conformality, while the second fixes the origin. The third is made in order to eliminate a certain redundancy: For each $\sigma > 0$, (δ, Γ) and $(\delta/\sigma, \sigma\Gamma)$ result in the same patch D. By fixing $\Gamma'(0) = 1$, we exclude all but $\sigma = 1$.

We also impose some symmetry properties for D. Specifically, we require that D is symmetric over the x_1-axis, which, in terms of Γ, is equivalent to

$$\operatorname{Re}\Gamma \text{ is odd in } x_1, \qquad \text{and} \qquad \operatorname{Im}\Gamma \text{ is even in } x_1.$$

Instead of using Γ, it is easier to work with a function defined only on the unit circle S^1. This is permissible because Γ can be recovered from the trace of its real part on S^1. Denote

$$\beta = \beta(\theta) := \operatorname{Re}[\Gamma(e^{i\theta}) - e^{i\theta}], \qquad \theta \in S^1,$$

then β determines Γ uniquely.

The assumptions on Γ in fact imply that the power series expansion of β has the following simple form:

$$\beta(\theta) = \sum_{n=2}^{\infty} \beta_n \cos\left(\frac{1}{2}(n-1)\pi + n\theta\right),$$

where $\beta_n \in \mathbb{R}$; the trivial solution corresponds to $\beta \equiv 0$.

Instead of using Γ, it is easier to work with a function defined only on the unit circle S^1. This is permissible because Γ can be recovered from the trace of its real part on S^1. Denote

$$\beta = \beta(\theta) := \operatorname{Re}[\Gamma(e^{i\theta}) - e^{i\theta}], \qquad \theta \in S^1,$$

then β determines Γ uniquely.

The assumptions on Γ in fact imply that the power series expansion of β has the following simple form:

$$\beta(\theta) = \sum_{n=2}^{\infty} \beta_n \cos\left(\frac{1}{2}(n-1)\pi + n\theta\right),$$

where $\beta_n \in \mathbb{R}$; the trivial solution corresponds to $\beta \equiv 0$.

Letting $\widetilde{F} := \widetilde{f} \circ \Gamma(\beta)$, the semi-linear problem for $\widetilde{f}$ becomes

$$\begin{cases} \Delta \widetilde{F} = |\partial_z \Gamma|^2 \gamma(\widetilde{F}) & \text{on } B_1 \\ \widetilde{F} = 0 & \text{on } S^1. \end{cases}$$

We have thus managed to fix the domain.

However, there is a technical point: we need in addition that $\int_{B_1} \Delta \widetilde{F}\, dx = 1$. With that in mind, we instead consider the problem

$$\begin{cases} \Delta \widetilde{F} = a|\partial_z \Gamma|^2 \gamma(\frac{1}{a}\widetilde{F}) & \text{on } B_1 \\ \widetilde{F} = 0 & \text{on } S^1 \\ \int_{B_1} \Delta \widetilde{F}\, dx = 1,\ a > 0. \end{cases}$$

By the assumptions on γ, for β in a neighborhood of 0, there exists a locally unique solution $(\widetilde{F}, a)$ to the above problem. Indeed, we show, with some effort, that it depends smoothly on β.

For each such β, then, the scaled vorticity $\widetilde{\omega}$ is determined:

$$\widetilde{\omega} \circ \Gamma = |\partial_z \Gamma|^{-2} \Delta \widetilde{F}.$$

Accordingly, we define the mapping

$$\widetilde{\Psi} : \beta \in X^s \mapsto \frac{1}{2\pi} a \int_{B_1} \log|\Gamma(\cdot) - \Gamma(z)| \gamma(\frac{1}{a} \widetilde{F}(z)) |\Gamma'(z)|^2 \, dz \in H^{s+1}(S^1)$$

where $\widetilde{F}$ is the unique solution to (depending on β), and the space

$$X^s := \{\beta \in H^s(S^1) : \beta = \sum_{n=2}^{\infty} \beta_n \cos(\frac{1}{2}(n-1)\pi + n\theta), \ \{\beta_n\}_{n=1}^{\infty} \subset \mathbb{R}\}.$$

We prove also that $\widetilde{\Psi}$ is smooth (of class C^2, say).

So what is $\widetilde{\Psi}$? Chasing definitions, we can see that it is the scaled trace on ∂D_0 of the rotational part of the relative stream function corresponding to the vortex patch. We will need to choose β appropriately so that we can take the solution to the semi-linear problem and find a corresponding relative stream function defined globally in Ω.

In other words, we must match the stream function found by solving the vortex dynamics, with that found in the boundary equation.

To do this, recall that we may decompose the relative stream function f into its rotational and irrotational parts as

$$\begin{aligned}\widetilde{f}(x) &= \frac{1}{\epsilon}\psi(\delta x) + \frac{c}{\epsilon}\delta x_2 \\ &= \widetilde{h}(\delta x) - \frac{1}{2\pi}\left(\log|\cdot| * \widetilde{\omega}\right)(x) + \frac{1}{2\pi}\log|\delta x - 2\mathbf{e}_2| + \delta\widetilde{c}x_2 + \mu\end{aligned}$$

where

$$\widetilde{h} := \frac{1}{\epsilon}h, \qquad \widetilde{c} := \frac{1}{\epsilon}c,$$

and μ is chosen so that the mean of f on ∂D is zero, which is permissible as φ determines h only up to a constant.

Note that h must be a complex conjugate of $\varphi_{\mathscr{H}}$, and hence this is is coupled to the boundary. The second log term is nothing but $\widetilde{\Psi} \circ \Gamma^{-1}$.

Since f has zero mean, we need only show that its trace on ∂D is a constant. Thus,

$$0 = \partial_\theta \left[\frac{1}{\epsilon} \mathscr{C}(\eta) \varphi_{\mathscr{H}} \circ \delta\Gamma(\beta) + \frac{c\delta}{\epsilon} \operatorname{Im}\Gamma(\beta) + \frac{1}{2\pi} \log|\delta\Gamma(\beta) - 2\mathbf{e}_2| + \widetilde{\Psi}(\beta) \right] |_{S^1}.$$

Here $\mathscr{C}$ denotes the harmonic conjugation operator. (Actually, this is just the Hilbert transform on the circle.)

We can show that for $\beta \in X^s$, the right-hand side above lies in Y^{s-1}, where

$$Y^s := \{\beta \in H^s(S^1) : \beta = \sum_{n=1}^{\infty} \beta_n \cos\left(\frac{1}{2}(n-1)\pi + n\theta\right), \ \{\beta_n\}_{n=1}^{\infty} \subset \mathbb{R}\}.$$

This suggests the following abstract formulation of the vortex patch problem: we seek

$$(\epsilon,\delta;\eta,c,\beta) \in \mathbb{R}^2 \times \mathscr{X},$$

such that

$$\mathscr{F}(\epsilon,\delta;\eta,c,\beta) = 0,$$

where $\mathscr{F} = (\mathscr{F}_1,\mathscr{F}_2,\mathscr{F}_3) : \mathbb{R}^2 \times \mathscr{X} \to \mathscr{Y}$, with

$$\mathscr{X} := H_{\mathrm{e}}^{s+1}(\mathbb{R}) \times (\dot{H}_{\mathrm{e}}^{s}(\mathbb{R}) \cap \dot{H}_{\mathrm{e}}^{1/2}(\mathbb{R})) \times \mathbb{R} \times X^s,$$
$$\mathscr{Y} := H_{\mathrm{e}}^{s-1}(\mathbb{R}) \times (\dot{H}_{\mathrm{e}}^{s-1}(\mathbb{R}) \cap \dot{H}_{\mathrm{e}}^{-1/2}(\mathbb{R})) \times Y^{s-1},$$

is given by

$$\mathscr{F}_1 = 0 \leftrightarrow \text{dynamic condition}, \qquad \mathscr{F}_2 = 0 \leftrightarrow \text{kinematic condition},$$

and

$$\mathscr{F}_3(\epsilon,\delta;\eta,\varphi,c,\beta) := \partial_\theta \Big[\frac{1}{\epsilon} \mathscr{C}(\eta)\varphi_{\mathscr{H}} \circ \delta\Gamma(\beta) + \frac{c\delta}{\epsilon} \operatorname{Im}\Gamma(\beta) + \frac{1}{2\pi} \log|\delta\Gamma(\beta) - 2\mathbf{e}_2| + \widetilde{\Psi}(\beta) \Big]\Big|_{S^1}.$$

Why is this good? Recall that the boundary problem (corresponding to $\mathscr{F}_1$ and $\mathscr{F}_2$) is easy:

$$D\mathscr{F}_1(0) = (g - \alpha^2 \partial_{x_1}^2, 0, 0, 0), \qquad D\mathscr{F}_2(0) = (0, \mathscr{G}(0), 0, 0),$$

where $D = (D_\eta, D_\varphi, D_c, D_\beta)$, hence,

$$D_\eta \mathscr{F}_1(0) : H_{\mathrm{e}}^{s+1}(\mathbb{R}) \to H_{\mathrm{e}}^{s-1}(\mathbb{R}),$$
$$D_\varphi \mathscr{F}_2(0) : \dot{H}_{\mathrm{e}}^{s}(\mathbb{R}) \cap \dot{H}_{\mathrm{e}}^{1/2}(\mathbb{R}) \to \dot{H}_{\mathrm{e}}^{s-1}(\mathbb{R}) \cap \dot{H}_{\mathrm{e}}^{-1/2}(\mathbb{R})$$

are invertible with compact inverse.

We can therefore solve for (η, φ) in terms of the other variables. The final equation $\mathscr{F}_3 = 0$, we use to select (β, c).

The challenging part here is to understand the operator $\partial_\theta \widetilde{\Psi}$, which recall arises from solving the scaled semi-linear elliptic problem.

$$\mathscr{F}_3(\epsilon, \delta; \eta, \varphi, c, \beta) := \partial_\theta \Big[\frac{1}{\epsilon} \mathscr{C}(\eta) \varphi_{\mathscr{H}} \circ \delta \Gamma(\beta) + \frac{c\delta}{\epsilon} \operatorname{Im} \Gamma(\beta) + \frac{1}{2\pi} \log |\delta \Gamma(\beta) - 2\mathbf{e}_2| + \widetilde{\Psi}(\beta) \Big] \Big|_{S^1}.$$

Miraculously, it turns out that

$$\text{null space } D_\beta \partial_\theta \widetilde{\Psi}(0) = \text{span}\,\{\theta \mapsto \sin(2\theta)\},$$

and in fact

$$D_\beta \partial_\theta \widetilde{\Psi}(0) \text{ is a Fredholm operator of index 0!}$$

The structure of the proof is then the following:

- Solve for (η, φ) in terms of $(\epsilon, \delta, c, \beta)$ using $\mathscr{F}_1, \mathscr{F}_2 = 0$,
- Use a Lyapunov–Schmidt like reduction to eliminate the null space of $D_\beta \partial_\theta \widetilde{\Psi}$,
- Use an implicit function theorem argument to solve for (c, β) in terms of (ϵ, δ) and a parameter arising from the projection.

The result is an existence theorem for the local bifurcation surface.

In fact, the asymptotic form of c is given by

$$c = \epsilon(\partial_{x_1}\varphi - \frac{1}{4\pi} + O(\delta)),$$

which recovers the correct formula for the vortex point.

Thanks for your attention