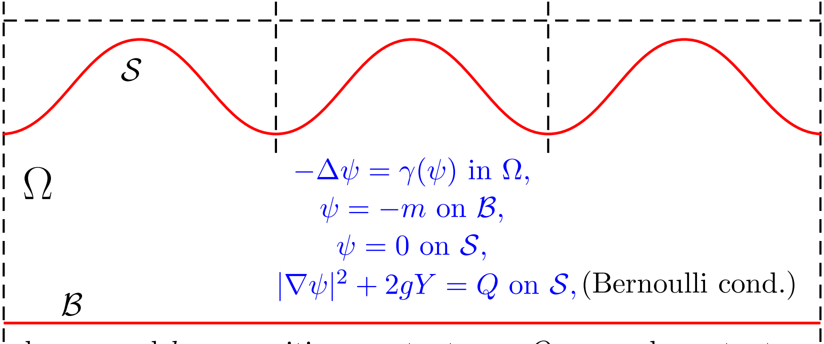
Variational formulations for steady water waves with constant vorticity

Eugen Varvaruca, University of Reading,
joint work with
Adrian Constantin, King's College London,
Walter Strauss, Brown University

The problem of steady gravity waves with vorticity

Find a $2\pi/k$-periodic curve $\mathcal{S}$ in the (X, Y)-plane, and a $2\pi/k$-periodic function ψ in the domain Ω between $\mathcal{S}$ and the real axis $\mathcal{B} = \{(X, 0) : X \in \mathbb{R}\}$, such that



$$
\begin{aligned}
-\Delta\psi &= \gamma(\psi) \text{ in } \Omega,\\
\psi &= -m \text{ on } \mathcal{B},\\
\psi &= 0 \text{ on } \mathcal{S},\\
|\nabla\psi|^2 + 2gY &= Q \text{ on } \mathcal{S}, \quad \text{(Bernoulli cond.)}
\end{aligned}
$$

where g and k are positive constants, m, Q are real constants, γ is a (given) function on the interval between $-m$ and 0.

We assume for most of the talk that $\gamma \equiv \text{constant}$.

Aims of the talk

- Reformulate the problem of **periodic steady gravity waves with constant vorticity γ in water of finite depth** as: find a 2π-periodic function v, with $[v] = h$, such that

$$\begin{aligned}\mathcal{C}_{kh}\left((Q-2gv)\,v'\right) + (Q-2gv)\left(\frac{1}{k} + \mathcal{C}_{kh}(v')\right)\\ - \frac{Q-2gh}{k} + 2g\,[v\,\mathcal{C}_{kh}(v')]\\ +2\gamma\,v\left(\frac{m}{kh} + \frac{\gamma}{2kh}[v^2] + \gamma\,\mathcal{C}_{kh}(vv')\right) - \frac{2\gamma m}{k}\\ -\gamma^2\mathcal{C}_{kh}(v^2v') - \gamma^2 v^2\left(\frac{1}{k} + \mathcal{C}_{kh}(v')\right) = 0,\end{aligned}$$

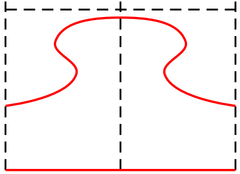
$$\begin{aligned}\left[\left\{\frac{m}{kh} + \gamma\left(\frac{[v^2]}{2kh} - \frac{v}{k} + \mathcal{C}_{kh}(vv') - v\mathcal{C}_{kh}(v')\right)\right\}^2\right]\\ = \left[(Q-2gv)\left\{v'^2 + \left(\frac{1}{k} + \mathcal{C}_{kh}(v')\right)^2\right\}\right],\end{aligned}$$

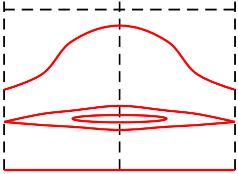
  where the wave has period $2\pi/k$, (conformal) mean depth h, flux m, vorticity γ, Bernoulli constant Q,
  $\mathcal{C}_{kh}$ is the 2π-periodic Hilbert transform associated to a strip of depth kh, and $[...]$ denotes average on a period.
- Variational structure of the above equation
- Bifurcation theory

Motivation: some interesting types of waves (with constant vorticity)

We are interested in the existence of the following types of waves:

- with overhanging profiles,
- with critical layers.





It is known that

- these waves cannot exist for $\gamma = 0$: Spielvogel(1973), Toland(2002), Shargorodsky&Toland (2008), V(2008);
- their existence is indicated by numerical computations for constant $\gamma \neq 0$: Simmen&Saffman(1985), Peregrine&Teles da Silva(1988), Vanden-Broeck(1994–1996), Okamoto&Shoji(2001), Ko&Strauss(2008–2009).

The existence of waves with constant vorticiy which have critical layers was conjectured by Kelvin(1880).

Fixed-domain reformulations

- **Nekrasov's formulation**: restricted to irrotational waves, very special, messy;
- **Babenko's formulation**: restricted to irrotational waves; advantages: involves a function of one variable (and the Hilbert transform), variational structure;

$$(1-2gv)\{v'^2+(1+\mathcal{C}v')^2\}=1, \qquad (\dagger)$$

$$\mathcal{C}((1-2\mu v)v')+(1-2\mu v)(1+\mathcal{C}v')=1. \qquad (*)$$

  $(*)$ is the *Euler-Lagrange equation* for

$$J(v)=\int_{-\pi}^{\pi} v\mathcal{C}v'-\mu v^2(1+\mathcal{C}v')\,dx.$$

- **Dubreil-Jacotin formulation**: valid for a general vorticity function γ; disadvantage: cannot handle either critical layers or overhanging profiles;
- **'flattening transformation' formulation**: valid for a general vorticity function γ; disadvantages: messy, cannot handle overhanging profiles, never been used successfully used in global bifurcation;
- **Dirichlet-Neumann operator formulation**: restricted to irrotational waves; disadvantage: never been used successfully in global bifurcation.

Waves with constant vorticity: generalization of (†)

Theorem (Constantin&V(2011))

Let (Ω, ψ) *be a* $2\pi/k$*-periodic water wave of class* $C^{1,\alpha}$*. Then there exists a constant* $h > 0$*, a function* $v \in C^{1,\alpha}_{2\pi}$ *and a constant* $a \in \mathbb{R}$ *such that* $[v] = h$*,*

$$\left\{ \frac{m}{kh} + \gamma \left(\frac{[v^2]}{2kh} - \frac{v}{k} + \mathcal{C}_{kh}(vv') - v\mathcal{C}_{kh}(v') \right) \right\}^2 = (Q - 2gv) \left\{ v'^2 + \left(\frac{1}{k} + \mathcal{C}_{kh}(v') \right)^2 \right\}.$$

and

$$\mathcal{S} = \left\{ \left(a + \frac{x}{k} + \mathcal{C}_{kh}(v - h)(x), v(x) \right) : x \in \mathbb{R} \right\}.$$

Conversely, let $h > 0$ *and* $v \in C^{1,\alpha}_{2\pi}$ *be such that the above equations hold, and let* $a \in \mathbb{R}$ *be arbitrary. Under the assumption that*

$$\mathbb{R} \ni x \mapsto \left(\frac{x}{k} + \mathcal{C}_{kh}(v - h)(x), v(x) \right) \text{ is injective,}$$

let $\mathcal{S}$ *be defined as above, and let* Ω *be the domain whose boundary consists of* $\mathcal{S}$ *and the real axis* $\mathcal{B}$*. Then there exists a function* ψ *in* Ω *such that* (Ω, ψ) *is a* $2\pi/k$*-periodic water wave of class* $C^{1,\alpha}$*.*

The new formulation: idea

Idea: regard the unknown fluid domain as the conformal image of a strip.

For any $d > 0$, let $\mathcal{R}_d$ be the strip

$$\mathcal{R}_d = \{(x, y) \in \mathbb{R}^2 : -d < y < 0\}.$$

For any domain Ω whose boundary consists of $\mathcal{B}$ and a L-periodic curve $\mathcal{S}$, we define its conformal mean depth as the unique positive constant h such that there exists a conformal mapping $U + iV$ from the strip $\mathcal{R}_h$ onto Ω (which extends as a homeomorphism between the closures of these domains), such that

$$U(x + L, y) = U(x, y) + L, \quad V(x + L, y) = V(x, y), \quad (x, y) \in \mathcal{R}_h.$$

If Ω has conformal mean depth h, and we write $L = 2\pi/k$, then one can regard Ω as the conformal image, through a mapping $U + iV$, of the strip $\mathcal{R}_{kh}$, where

$$U(x + 2\pi, y) = U(x, y) + \frac{2\pi}{k}, \quad V(x + 2\pi, y) = V(x, y), \quad (x, y) \in \mathcal{R}_{kh}.$$

Equation $\Delta\psi = -\gamma$ in Ω is equivalent to

$$(X,Y) \mapsto \psi(X,Y) + \frac{\gamma}{2}Y^2 \quad \text{is a harmonic function in } \Omega.$$

Let $\zeta : \mathcal{R}_{kh} \to \mathbb{R}$ be given by

$$\zeta(x,y) := \psi(U(x,y), V(x,y)) + \frac{\gamma}{2}V^2(x,y) + m.$$

Then

$$\begin{aligned}
&\Delta\zeta = 0 \quad \text{in } \mathcal{R}_{kh},\\
&\zeta(x,-kh) = 0 \quad \text{for all } x \in \mathbb{R},\\
&\zeta(x,0) = m + \frac{\gamma}{2}V^2(x,0) \quad \text{for all } x \in \mathbb{R},\\
&(\zeta_y - \gamma V V_y)^2 = (Q - 2gV)(V_x^2 + V_y^2) \quad \text{at } (x,0) \text{ for all } x \in \mathbb{R}.
\end{aligned}$$

The periodic Hilbert transform for a strip

Let $\mathcal{R}_d = \{(x,y) : x \in \mathbb{R}, -d < y < 0\}$. Let w be 2π-periodic and of zero mean, and let $W : \overline{\mathcal{R}}_d \to \mathbb{R}$ be the unique solution of

$$\Delta W = 0 \text{ in } \mathcal{R}_d, W(x,-d) = 0,\ x \in \mathbb{R}, W(x,0) = w(x),\ x \in \mathbb{R}.$$

Let Z be such that $Z + iW$ is holomorphic in $\mathcal{R}_d$ (and Z is 2π-periodic). We define the Hilbert transform $\mathcal{C}_d(w)$ by:
$\mathcal{C}_d(w)(x) := Z(x,0),\ x \in \mathbb{R}$, (normalized: of zero mean.)
Then $\mathcal{C}_d$ is a linear operator satisfying, for all $n \in \mathbb{N}$,

$$\mathcal{C}_d(\cos(nx)) = \coth(nd)\sin(nx), \quad \mathcal{C}_d(\sin(nx)) = -\coth(nd)\cos(nx).$$

Formally, for $d = \infty$, we get the standard Hilbert transform

$$\mathcal{C}(w)(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cot((t-s)/2)w(s)\,ds.$$

The Cauchy-Riemann equations show that the Dirichlet-Neumann operator of a periodic function is

$$w \mapsto \frac{[w]}{d} + \mathcal{C}_d(w').$$

The new formulation: revisited

Let (Ω, ψ) be a $2\pi/k$-periodic water wave, and let h be the conformal mean depth of Ω. Recall that

$$U(x+2\pi, y) = U(x,y) + \frac{2\pi}{k}, \quad V(x+2\pi, y) = V(x,y), \quad (x,y) \in \mathcal{R}_{kh},$$

and

$$\begin{aligned}
&\Delta\zeta = 0 \quad \text{in } \mathcal{R}_{kh},\\
&\zeta(x, -kh) = 0 \quad \text{for all } x \in \mathbb{R},\\
&\zeta(x,0) = m + \frac{\gamma}{2}V^2(x,0) \quad \text{for all } x \in \mathbb{R},\\
&(\zeta_y - \gamma V V_y)^2 = (Q - 2gV)(V_x^2 + V_y^2) \quad \text{at } (x,0) \text{ for all } x \in \mathbb{R}.
\end{aligned}$$

Let $v(x) := V(x,0)$. Then $[v] = h$,

$$\mathcal{S} = \left\{ \left(a + \frac{x}{k} + \mathcal{C}_{kh}(v-h)(x), v(x) \right) : x \in \mathbb{R} \right\},$$

and

$$\left\{ \frac{m}{kh} + \gamma\frac{[v^2]}{2kh} + \gamma\mathcal{C}_{kh}(vv') - \gamma v \left(\frac{1}{k} + \mathcal{C}_{kh}(v') \right) \right\}^2 = (Q - 2gv) \left\{ v'^2 + \left(\frac{1}{k} + \mathcal{C}_{kh}(v') \right)^2 \right\}.$$

Local bifurcation

Theorem (Constantin&V(2011))

Given $h > 0$, $k > 0$ and $\gamma \in \mathbb{R}$, for any $m \in \mathbb{R}$ there exists a laminar flow with a flat free surface in water of depth h, of constant vorticity γ and relative mass flux m. Moreover, the values $m_\pm$ of the flux given by

$$m_\pm = \frac{\gamma h^2}{2} - \frac{\gamma h \tanh(kh)}{2k} \pm h \sqrt{\frac{\gamma^2 \tanh^2(kh)}{4k^2} + g \frac{\tanh(kh)}{k}}$$

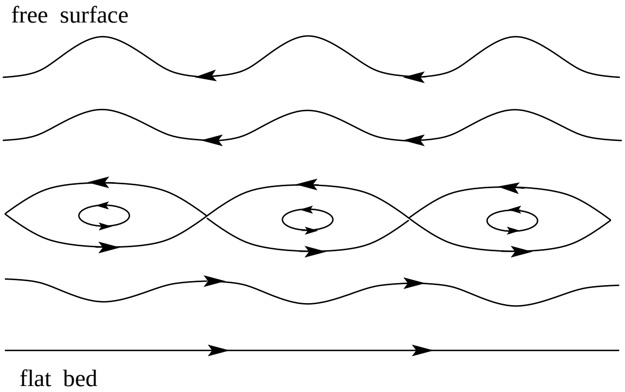
trigger the appearance of steady periodic waves (symmetric and monotone) of small amplitude, with period $2\pi/k$ and conformal mean depth h. The laminar flows of flux $m_\pm$ are exactly those with horizontal speeds at the flat free surface $\lambda_\pm$ given by

$$\lambda_\pm = -\frac{\gamma \tanh(kh)}{2k} \pm \sqrt{\frac{\gamma^2 \tanh^2(kh)}{4k^2} + g \frac{\tanh(kh)}{k}}.$$

The above dispersion relation was obtained by formal arguments by Thompson (1949), Biesel (1950). A rigorous proof of existence of these small-amplitude nonlinear waves was first obtained by Wahlén (2009). Our own proof relies, in the usual way, on Crandall-Rabinowitz local bifurcation theorem.

Local bifurcation

For $\gamma \neq 0$, some of these bifurcation-inducing laminar flows have a line of stagnation points (never at the free surface). The corresponding bifurcating waves have critical layers.



Theorem (Rabinowitz global bifurcation theorem, (1973))

Let X be a Banach space and $F \in C^1(\mathbb{R} \times X, X)$. Suppose that $F(\lambda, u) = u - G(\lambda, u)$ where $G : \mathbb{R} \times X \to X$ is a nonlinear compact operator, $G(\lambda, 0) = 0$ for all $\lambda \in \mathbb{R}$, and $G(\lambda, u) = \lambda L u + H(\lambda, u)$, where $H(\lambda, u) = o(||u||)$ for u near 0 uniformly on bounded λ intervals, and L is a compact linear operator on X.
Suppose also that λ^* is a characteristic value of L (i.e. there exists $u^* \in X \setminus \{0\}$ such that $u^* = \lambda^* L u^*$) of odd multiplicity.
Then the set of nontrivial solutions of $F(\lambda, u) = 0$ has a connected component $\mathcal{M}$ which contains λ^* in its closure, and either
(I) is unbounded in $\mathbb{R} \times X$, or
(II) contains in its closure a point $(\mu^*, 0)$, where $\mu^* \neq \lambda^*$ is another characteristic value of L.

Remark. If F is as above, but defined only on an open set $\mathcal{O}$ of $\mathbb{R} \times X$, then a similar conclusion holds, except that now a third alternative may hold for $\mathcal{M}$:
(III) $\mathcal{M}$ contains a sequence (λ_n, u_n) approaching the boundary of $\mathcal{O}$.

The new formulation

Theorem (Constantin&Strauss&V(2012))

The previous equation can be equivalently rewritten as: $[v] = h$ *and*

$$\mathcal{C}_{kh}\left(\left(Q - 2gv - \gamma^2 v^2\right) v'\right) + \left(Q - 2gv - \gamma^2 v^2\right)\left(\frac{1}{k} + \mathcal{C}_{kh}(v')\right)$$
$$+ 2\gamma\, v\left(\frac{m}{kh} + \frac{\gamma}{2kh}[v^2] + \gamma\, \mathcal{C}_{kh}(vv')\right)$$
$$- \frac{Q + 2\gamma m - 2gh}{k} + 2g\left[v\, \mathcal{C}_{kh}(v')\right] = 0,$$

$$\left[\left\{\frac{m}{kh} + \gamma\Big(\frac{[v^2]}{2kh} - \frac{v}{k} + \mathcal{C}_{kh}(vv') - v\mathcal{C}_{kh}(v')\Big)\right\}^2\right]$$
$$= \left[(Q - 2gv)\left\{v'^2 + \left(\frac{1}{k} + \mathcal{C}_{kh}(v')\right)^2\right\}\right].$$

We have a proof of this result using Riemann-Hilbert theory.
For $\gamma = 0$ and $h = \infty$: Babenko(1987), Buffoni, Dancer& Toland(2000), Shargorodsky&Toland(2003–2008).

Variational formulation in the physical plane

Consider the functional

$$\mathcal{L}(\Omega, \psi) = \iint_{\Omega^\dagger} \left\{ |\nabla\psi|^2 - 2\gamma\psi + Q - 2gY \right\} d\mathbb{X}.$$

The domain of definition of $\mathcal{L}$ is the space

$$\begin{aligned}\mathcal{A} := \{(\Omega, \psi) : \Omega \text{ bounded below by the real axis } \mathcal{B} \\ \text{and above by a (smooth) Jordan curve } \mathcal{S} \\ \text{that is } 2\pi/k\text{-periodic in the horizontal direction,} \\ \psi : \Omega \to \mathbb{R} \text{ satisfies } \psi = 0 \text{ on } \mathcal{S} \text{ and } \psi = -m \text{ on } \mathcal{B}.\}\end{aligned}$$

Here $\Omega^\dagger$ denotes a period of Ω, and $\mathbb{X} = (X, Y)$.

Theorem (?? Friedrichs(1933))

Any critical point (Ω, ψ) of the functional $\mathcal{L}$ over the space $\mathcal{A}$ is a solution to the steady water wave problem.

A similar result is valid, with appropriate modifications, for any vorticity function γ.

Variational formulation in the physical plane

Standard variations of ψ, with Ω fixed:

$$\frac{d}{d\varepsilon}\mathcal{L}(\Omega, \psi + \varepsilon\phi) = 0 \text{ for all } \phi \in C_0^1(\Omega) \quad \Rightarrow \quad \Delta\psi = -\gamma.$$

Inner variations: Take $\Phi_\varepsilon(\mathbb{X}) = \mathbb{X} + \varepsilon\Phi(\mathbb{X})$, where $\Phi \in C_0^1(\mathbb{R}_+^2; \mathbb{R}^2)$ (in fact, periodic in the horizontal direction, and with support which is bounded in the vertical direction). For ε sufficiently small, Φ_ε is a diffeomorphism between Ω and a domain $\Omega_\varepsilon = \Phi_\varepsilon(\Omega)$ bounded by $\mathcal{B}$ and a periodic curve $\mathcal{S}_\varepsilon$. In Ω_ε define a function ψ_ε by $\psi_\varepsilon = \psi(\Phi_\varepsilon^{-1})$. Then $(\Omega_\varepsilon, \psi_\varepsilon) \in \mathcal{A}$, and

$$\frac{d}{d\varepsilon}\mathcal{L}(\Omega_\varepsilon, \psi_\varepsilon) = 0 \text{ for all } \Phi \in C_0^1(\mathbb{R}_+^2; \mathbb{R}^2)$$

implies the Bernoulli condition $|\nabla\psi|^2 + 2gY = Q$.

Variational formulation in the conformal plane

For any Ω, let ψ_Ω be the unique solution of

$$\psi = 0 \text{ on } \mathcal{S},\ \psi = -m \text{ on } \mathcal{B},\ \Delta\psi = -\gamma \text{ in } \Omega.$$

Let $U + iV$ be the conformal mapping from $\mathcal{R}_{kh}$ to Ω, so that

$$\mathcal{S} = \left\{ \left(a + \frac{x}{k} + \mathcal{C}_{kh}(v - h)(x), v(x) \right) : x \in \mathbb{R} \right\},$$

where $v(x) = V(x, 0)$ for all $x \in \mathbb{R}$. Then

$$\begin{aligned}
\mathcal{L}(\Omega, \psi_\Omega) = \int_{-\pi}^{\pi} \left(-\frac{\gamma^2}{3} v^3 - g\, v^2 + Q\, v \right) \left(\frac{1}{k} + \mathcal{C}_{kh}(v') \right) dx \\
+ \int_{-\pi}^{\pi} \left(m - \frac{\gamma}{2} v^2 \right) \left(\frac{m}{kh} - \frac{\gamma}{2kh} [v^2] - \gamma\, \mathcal{C}_{kh}(vv') \right) dx \\
=: \Lambda(v).
\end{aligned}$$

Variational formulation in the conformal plane

Theorem (Constantin&Strauss&V(2012))

Any critical point v with $[v] = h$ of the functional Λ on the previous slide satisfies

$$\mathcal{C}_{kh}\left(\left(Q - 2gv - \gamma^2 v^2\right) v'\right) + \left(Q - 2gv - \gamma^2 v^2\right)\left(\frac{1}{k} + \mathcal{C}_{kh}(v')\right) + 2\gamma\, v\left(\frac{m}{kh} + \frac{\gamma}{2kh}[v^2] + \gamma\, \mathcal{C}_{kh}(vv')\right) - \frac{Q + 2\gamma m - 2gh}{k} + 2g\left[v\, \mathcal{C}_{kh}(v')\right] = 0,$$

$$\left[\left\{\frac{m}{kh} + \gamma\left(\frac{[v^2]}{2kh} - \frac{v}{k} + \mathcal{C}_{kh}(vv') - v\mathcal{C}_{kh}(v')\right)\right\}^2\right] = \left[(Q - 2gv)\left\{v'^2 + \left(\frac{1}{k} + \mathcal{C}_{kh}(v')\right)^2\right\}\right].$$

The proof is a 4 page calculation. We write $v = w + h$, where $[w] = 0$. In the proof we make use of $\frac{d}{dh}\Lambda(w + h) = 0$, which requires the nice and unexpected identity (for any 2π-periodic function f)

$$\frac{d}{dh}\left(\mathcal{C}_{kh}\left(f'\right)\right) = -\,k\, f'' - \,k\, \mathcal{C}_{kh}^2\left(f''\right).$$

Global bifurcation: compactness

- The first equation can be rearranged as:

$$\begin{aligned}(Q-2gv)\mathcal{C}_{kh}(v') = {} & (g-\gamma^2 v)(\mathcal{C}_{kh}(vv') - v\mathcal{C}_{kh}(v')) \\ & + \frac{\gamma^2}{2}(\mathcal{C}_{kh}(v^2v') - v^2\mathcal{C}_{kh}(v')) \\ & + \frac{\gamma^2}{2kh}(hv^2 - [v^2]v) + \left(\frac{g}{k} - \frac{\gamma m}{kh}\right)(v-h) - g[v\mathcal{C}_{kh}(v')].\end{aligned}$$

- Commutator estimates: $v \in C^{n,\alpha}_{2\pi}$ implies that $\mathcal{C}_{kh}(vv') - v\mathcal{C}_{kh}(v'), \mathcal{C}_{kh}(v^2v') - v^2\mathcal{C}_{kh}(v') \in C^{n,\delta}_{2\pi}$ for all $\delta \in (0,\alpha)$ (and not merely $\in C^{n-1,\alpha}_{2\pi}$).
- Applicability of Rabinowitz Theorem: The commutator estimates imply that, upon dividing by $Q-2gv$, adding v to both sides, and inverting the operator $v \mapsto v + \mathcal{C}_{kh}(v')$, the equation can be put in the form $v = \mathcal{A}(m, Q, v)$, where $\mathcal{A}$ is a nonlinear compact operator on $C^{n,\alpha}_{2\pi}$ for any $n \geq 1$ and $\alpha \in (0,1)$.
- Regularity: A byproduct of the above ideas is that $v \in C^{1,\alpha}_{2\pi}$ and $Q - 2gv > 0$ (no stagnation points) implies $v \in C^{\infty}_{2\pi}$.

Global bifurcation

Rabinowitz Theorem gives the existence of a connected set of solutions extending the local bifurcation curve and containing a sequence (m_n, Q_n, v_n) such that either:

- $(m_n, Q_n, v_n) \to \infty$ in $\mathbb{R} \times \mathbb{R} \times C^{1,\alpha}_{2\pi}$,
- $\min\{Q_n - 2gv_n(x) : x \in \mathbb{R}\} \to 0$ as $n \to \infty$ (stagnation at the crest).

(A third possible alternative, that the set of solutions returns to the line of trivial solutions (flat laminar flows) is ruled out in the standard way, using preservation of the nodal pattern along the continuum: v is even, $v' < 0$ on $(0, \pi)$, $v''(0) < 0 < v''(\pi)$.)
The solutions that we construct give rise to water waves if and only if

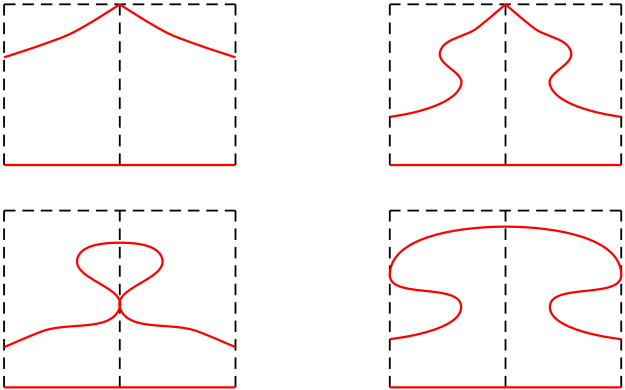
the curve $\mathcal{S} = \left\{ \left(\frac{x}{k} + \mathcal{C}_{kh}(v - h)(x), v(x) \right) : x \in \mathbb{R} \right\}$ is non-self-intersecting.

So, it is possible in principle that at some point along our solution set, the non-self-intersection property fails.

Global bifurcation: conjectures

We expect that the connected set of solutions can be continued until one of the following situations is reached:

- extreme waves with stagnation points and corners of 120° at the crests, whose profile is either a graph or overhanging;
- overhanging waves with self-intersections: two possible situations.



Thank you for your attention!