On integral properties of steady gravity waves on water of finite depth

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Topics

- Statement of the problem
- Various operator equations (a review)
- Equivalent statement of the problem
- Integral properties of bounded steady waves

1:

Statement of the problem for waves of general form

Two-dimensional irrotational water waves
in a horizontal open channel
of uniform rectangular cross-section
with a flat rigid bottom

- Water is bounded above by the free surface that does not touch the bottom.
- Cartesian coordinates (X, Y) are such that the bottom coincides with the X-axis and gravity acts in the negative Y-direction; g denotes the acceleration due to gravity.
- The frame of reference is chosen so that the velocity field and the free surface are time-independent.
- The free surface is the graph of $Y = \xi(X)$, where $\xi(X) > 0$ for all $X \in \mathbb{R}$, $\Rightarrow$ $\mathcal{D} = \{X \in \mathbb{R},\ 0 < Y < \xi(X)\}$ is the longitudinal section of the water domain.

Statement of the problem

- The water motion is irrotational and two-dimensional

$\Rightarrow$ there exists a stream function $\Psi(X, Y)$.

Constant density of water + irrotational motion $\Rightarrow$

$$\nabla^2 \Psi = 0 \quad \text{in the water domain } \mathcal{D}.$$

No normal flow on the bottom

$$\Rightarrow \quad \Psi(X, 0) = 0, \;\; X \in \mathbb{R}.$$

The kinematic condition on the free surface

$$\Rightarrow \quad \Psi(X, \xi(X)) = Q, \;\; X \in \mathbb{R}.$$

Bernoulli's eq. on the free surface (the surface tension neglected)

$$\Rightarrow \quad \frac{1}{2}|\nabla\Psi|^2 + g\xi = R, \quad Y = \xi(X), \;\; X \in \mathbb{R}.$$

- $Q > 0$ is the volume rate of flow per unit span.
- $R > 0$ is the total head (Bernoulli's constant).

2:

Various operator equations for steady waves
(a review)

2a: General steady waves

The Benjamin–Lighthill conjecture (1954)

The parameters Q, R and S (the latter is referred to as the flow force) determine any steady wave-train.

Benjamin (JFM, 1995): "Specifically, in terms of [the dimensionless] parameters

$$r = \frac{R}{R_c} \text{ and } s = \frac{S}{S_c}, \quad R_c = \frac{3}{2}(Qg)^{2/3}, \quad S_c = \frac{3}{2}(Q^4 g)^{1/3},$$

such waves [. . .] realize points (r, s) inside the region of the (r, s)-plane that is bounded by the cusped curve representing all possible uniform streams."

Let $\xi(X) = \xi_u = \text{const} \;\; \Rightarrow \;\;$ Bernoulli's eq.:

$$\left(\frac{Q}{\xi_u}\right)^2 + 2g\xi_u = 2R, \quad \xi_u \in \mathbb{R}.$$

For every $Q > 0$ and every $R > R_c$ there exist two positive roots ξ_- and ξ_+ such that $\xi_- < \xi_c < \xi_+$, where $\xi_c = (Q^2/g)^{1/3}$ is the only double root corresponding to $R = R_c$.

The two stream solutions are as follows:

$$\left(\frac{Q}{\xi_\pm}Y, \, \xi_\pm\right).$$

The *subcritical* (*supercritical*) flow corresponds to the $+$ $(-)$ sign.

Dimensionless variables for general waves

For $R \geqslant R_c$ we put

$$x = \frac{X}{\xi_-},\ y = \frac{Y}{\xi_-} - 1; \quad \eta(x) = \frac{\xi(X)}{\xi_-} - 1; \quad \psi(x, y) = \frac{\Psi(X, Y)}{Q}.$$

Then the problem of steady waves takes the form:

$$\begin{aligned}
&\psi_{xx} + \psi_{yy} = 0, \quad (x, y) \in D = \{x \in \mathbb{R}, -1 < y < \eta(x)\};\\
&\psi = 0, \quad y = -1,\ x \in \mathbb{R};\\
&\psi = 1, \quad y = \eta(x),\ x \in \mathbb{R};\\
&|\nabla\psi|^2 + 2\lambda\eta = 1, \quad y = \eta(x),\ x \in \mathbb{R} \setminus \Sigma_\eta;
\end{aligned}$$

$\Sigma_\eta = \{x \in \mathbb{R} : 1 - 2\lambda\eta(x) = 0\}$ is the set of stagnation points;

$$\lambda = \frac{g\xi_-^3}{Q^2} = \left(\frac{\xi_-}{\xi_c}\right)^3 \in (0, 1].$$

Bernoulli's constant (the total head):

$$r = \frac{1 + 2\lambda}{3\lambda^{2/3}} \quad \text{for } \lambda \leqslant 1.$$

The flow force:

$$s = \frac{1}{3\lambda^{1/3}} \left[1 + \lambda + \eta(x) - \lambda\eta^2(x) + \int_{-1}^{\eta(x)} \left(\psi_y^2 - \psi_x^2\right) \, \mathrm{d}y \right].$$

The flow force for supercritical uniform streams:

$$s_-(\lambda) = \frac{2 + \lambda}{3\lambda^{1/3}} \text{ for } \lambda \leqslant 1.$$

The flow force for subcritical uniform streams:

$$s_+(\lambda) > s_-(\lambda) \text{ for } \lambda < 1.$$

Integro-differential equations for general waves

In two papers published in Arch. Rat. Mech. Anal. (2010, 2011), the following equations are used for proving the Benjamin–Lighthill conjecture for λ close to unity.

After application of the hodograph transform, the problem equivalently reduces to the equation (2010):

$$(N\eta)(\phi) = \left[\frac{1}{1 - 2\lambda\eta(\phi)} - \eta_\phi^2(\phi)\right]^{1/2} - 1, \quad \phi \in \mathbb{R} \setminus \Sigma_y.$$

Here $(Nf)(\phi) = \left(F^{-1}_{\tau\mapsto\phi}\, \tau \coth \tau\, F_{\varphi\mapsto\tau}\right)[f(\varphi)]$, and Σ_y is the image of the set of stagnation points Σ_η on the physical plane.

Another equation that arises after the hodograph transform is as follows (2011):

$$(1 - 2\lambda\eta)N\eta' = 2\lambda\eta'(1 + N\eta) - N[\eta'(1 - 2\lambda\eta)].$$

2b:

Operator equations for Stokes waves
on water of finite depth

Okamoto's equation for Stokes waves

In (1973), Zeidler reduced the problem of Stokes waves on water of finite depth to an operator equation (see his *Functional Analysis*, IV, ch. 71).

Okamoto (1990) obtained more explicit equation of similar form:

$$e^{3H_\rho\theta}\,\frac{\mathrm{d}H_\rho\theta}{\mathrm{d}\sigma} = \frac{g\ell}{\pi C^2}\sin\theta.$$

Here $\sigma \in [0, 2\pi]$ is an auxiliary variable through which one expresses the unknown angle θ between the free surface and the x-axis; 2ℓ is the wavelength, C is the velocity of wave propagation and $\rho \in [0, 1)$ is a scaling parameter.

Okamoto's equation for Stokes waves (continued)

The operator H_ρ is defined as follows:

$$\begin{aligned}
&H_\rho\left(\sum_{n=1}^{\infty}(a_n \sin n\sigma + b_n \cos n\sigma)\right) \\
&= \sum_{n=1}^{\infty} \frac{1+\rho^{2n}}{1-\rho^{2n}}(-a_n \cos n\sigma + b_n \sin n\sigma) \\
&= H_0\left(\sum_{n=1}^{\infty}(a_n \sin n\sigma + b_n \cos n\sigma)\right) \\
&+2\sum_{n=1}^{\infty} \frac{\rho^{2n}}{1-\rho^{2n}}(-a_n \cos n\sigma + b_n \sin n\sigma),
\end{aligned}$$

where H_0 is the Hilbert transform on $(0, 2\pi)$.

Dimensionless variables for Stokes waves

Let ξ be even, $2\,\ell$-periodic function, and let Ψ have the same properties in X.

We put $H = \frac{1}{2\,\ell} \int_{-\ell}^{\ell} \xi(X)\,\mathrm{d}X$ (the unknown mean depth),

$$x = \tfrac{\pi}{\ell} X, \quad y = \tfrac{\pi}{\ell}(Y - H); \quad \eta(x) = \tfrac{\pi}{\ell}[\,\xi(X) - H\,],$$

and $\psi(x,y) = [\,\Psi(X,Y) - Q\,]/Q_0$, where $Q_0 = \left[g(\ell/\pi)^3\right]^{1/2}$.

$\Rightarrow$ The problem of Stokes waves takes the form:

$$\psi_{xx} + \psi_{yy} = 0 \text{ in } D_S = \{x \in (-\pi,\pi); -\pi H/\ell < y < \eta(x)\};$$

$$\psi = -Q/Q_0, \quad y = -\pi H/\ell, \; x \in (-\pi,\pi);$$

$$\psi = 0, \quad y = \eta(x), \; x \in (-\pi,\pi), \text{ where } \int_{-\pi}^{\pi} \eta(x)\,\mathrm{d}x = 0;$$

$$|\nabla\psi|^2 + 2\eta = \mu, \quad y = \eta(x), \; x \in (-\pi,\pi).$$

The dimensionless problem's parameters

- Q/Q_0 is the given normalized rate of flow.
- $\mu = 2\pi(R - gH)/g\ell$ is the unknown parameter.

What is the sense of μ ?

If the depth is infinite, then $\mu = \pi C^2/g\ell$ is the unknown Froude number squared.

In the present case, it is easy to show that $\mu = \pi c^2/g\ell$, where

$$c^2 = \frac{1}{2\,\ell}\int_{-\ell}^{\ell} |\,\nabla\Psi(X,\,\xi(X))\,|^2\,\mathrm{d}X = \frac{1}{2\,\ell}\int_{-\ell}^{\ell} \left|\frac{\partial\Psi}{\partial n}(X,\,\xi(X))\right|^2\mathrm{d}X$$

is the velocity on the free surface squared and averaged. Does $c = C$?

The problem of Stokes waves is equivalent to the following single equation:

$$\mu\left[\mathcal{B}_r(\eta')\right](t) = \eta(t) + \eta(t)\left[\mathcal{B}_r(\eta')\right](t) + \left[\mathcal{B}_r(\eta'\eta)\right](t). \quad (*)$$

Here $t \in [-\pi, \pi]$ is an auxiliary variable through which one expresses x and η;

$\mathcal{B}_r = \mathcal{C} + \mathcal{K}_r$, where $\mathcal{C}$ is the Hilbert transform on $(-\pi, \pi)$ and

$$(\mathcal{K}_r f)(t) = \pi^{-1} \int_{-\pi}^{\pi} f(\tau)\, K_r(t-\tau)\, \mathrm{d}\tau,$$

$$K_r(t-\tau) = 2\sum_{k=1}^{\infty} \frac{r^{2k}}{1-r^{2k}} \sin k(t-\tau). \quad (**)$$

Babenko's equation for Stokes waves (continued)

In $(**)$, $r \in (0,1)$ is a scaling parameter related to Q/Q_0; in particular, one can take $r = \exp\{-Q/Q_0\}$.

Equation $(*)$ has the same form as (1.1) in

[1] B. Buffoni, E. N. Dancer, J. F. Toland,

The sub-harmonic bifurcation of Stokes waves. *Arch. Ration. Mech. Anal.* **152** (2000), 241–271.

The only difference is that $\mathcal{B}_r$ stands in $(*)$ instead of $\mathcal{C}$ in (1.1), [1].

3:

Another statement of the problem for general waves

Statement of the problem for the velocity potential

Let ϕ be a harmonic conjugate to ψ in D
$\Rightarrow$ the problem of steady waves:

$$\phi_{xx} + \phi_{yy} = 0, \quad (x,y) \in D = \{x \in \mathbb{R}, -1 < y < \eta(x)\};$$
$$\phi_y = 0, \quad y = -1, \ x \in \mathbb{R};$$
$$\phi_y = \eta_x \phi_x, \quad y = \eta(x), \ x \in \mathbb{R} \setminus \Sigma_\eta;$$
$$|\nabla \phi|^2 + 2\lambda\eta = 1, \quad y = \eta(x), \ x \in \mathbb{R} \setminus \Sigma_\eta.$$

However, it must be complemented by the auxiliary condition

$$\int_{-1}^{\eta(x)} \phi_x(x,y)\,\mathrm{d}y = 1$$

assumed to be valid for some $x \in \mathbb{R}$, and so holding for all $x \in \mathbb{R}$ by the first Green's formula.

Let $v(x,y) = \phi(x,y) - u(x)$, where

$$u(x) = \frac{1}{1+\eta(x)} \int_{-1}^{\eta(x)} \phi(x,y)\,\mathrm{d}y.$$

The following properties are obvious:
(1) v remains the same when a constant is added to ϕ;
(2) v vanishes identically for uniform streams;
(3) $\int_{-1}^{\eta(x)} v(x,y)\,\mathrm{d}y = 0$.

Key relations

(i) $\phi_y = \eta_x \phi_x$ when $y = \eta(x) \Rightarrow$ Bernoulli's equation:

$$\text{(ii)} \quad \left(1+\eta_x^2\right)(v_x + u_x)^2 + 2\lambda\eta = 1, \quad y = \eta(x),\ x \in \mathbb{R},$$

$$\text{and (iii) } u_x(x) = \frac{1+\eta_x(x)v(x,\eta(x))}{1+\eta(x)} \Leftarrow \int_{-1}^{\eta(x)} \phi_x(x,y)\,\mathrm{d}y = 1.$$

$$\text{(i) and (iii)} \Rightarrow \text{(iv) } v_y = \eta_x\left(v_x + \frac{1+\eta_x v}{1+\eta}\right), \quad y = \eta(x),\ x \in \mathbb{R} \Rightarrow$$

$$\text{(v) } \left[(1+\eta)\,v(x,\eta(x))\right]_x = \left(1+\eta_x^2\right)\left(1 + v_x + \eta_x v + \eta v_x\right) - 1.$$

Equivalent operator equation

Thus, (i)–(v) reduce the problem of steady waves to a single equation for η:

$$[(1+\eta)\,v(x,\eta(x))]_x = (1+\eta)\sqrt{(1+\eta_x^2)\,(1-2\lambda\eta)} - 1 \quad (***)$$

valid for a.e. $x \in \mathbb{R}$. Here v is considered as the image of the nonlinear mapping $\eta \mapsto v$ defined by the integral identity:

$$\int_D \nabla\zeta\cdot\nabla v\,\mathrm{d}x\mathrm{d}y = \int_{-\infty}^{+\infty} \zeta(x,\eta(x))\,\frac{\eta_x(x)\,[1+\eta_x(x)\,v(x,\eta(x))]}{1+\eta(x)}\,\mathrm{d}x,$$

which must hold for an arbitrary smooth ζ having a compact support in $\bar{D}$ and satisfying the orthogonality condition (3).

4:

Integral properties of bounded steady waves

Integral properties of bounded steady waves

Theorem (J. Math. Fluid Mech., 2009)

Let v be the function defined above. Then $\sup_{(x,y)\in D} |v(x,y)| < \infty$.

Integrating $(***)$ over an interval $(x_-, x_+) \subset \mathbb{R}$, we obtain

$$[(1+\eta(x))\, v(x,\eta(x))]_{x=x_-}^{x=x_+} = \int_{x_-}^{x_+} \left[(1+\eta)\sqrt{(1+\eta_x^2)(1-2\lambda\eta)} - 1\right] \mathrm{d}x.$$

Corollary 1 (J. Math. Fluid Mech., 2009)

Every bounded steady wave profile η satisfies the following integral property:

$$\lim_{x_+ - x_- \to \infty} \frac{1}{x_+ - x_-} \int_{x_-}^{x_+} (1+\eta)\sqrt{(1+\eta_x^2)(1-2\lambda\eta)}\, \mathrm{d}x = 1.$$

Integral properties of bounded steady waves (continued)

Corollary 2 (J. Math. Fluid Mech., 2009)

(i) If the wave profile η is periodic and if $x_+ - x_-$ is an integer non-zero multiple of the wavelength, then

$$\frac{1}{x_+ - x_-}\int_{x_-}^{x_+} (1+\eta)\sqrt{(1+\eta_x^2)(1-2\lambda\eta)}\,\mathrm{d}x = 1.$$

(ii) If η is the profile of a solitary wave, then

$$\int_{-\infty}^{+\infty}\left[(1+\eta)\sqrt{(1+\eta_x^2)(1-2\lambda\eta)} - 1\right]\mathrm{d}x = 0.$$

Thank you . . .