Stokes waves on vortical flows with counter-currents

Vladimir Kozlov

Joint work with N. Kuznetsov

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Let

$$\mathcal{D} = \{-\infty < X < +\infty,\ 0 < Y < \xi(X)\}.$$

Since the surface tension is neglected, Ψ and ξ satisfy

$$\begin{aligned}
&\Psi_{XX} + \Psi_{YY} + \omega(\Psi) = 0, \quad (X, Y) \in \mathcal{D};\\
&\Psi(X, 0) = 0, \quad X \in \mathbb{R};\\
&\Psi(X, \xi(X)) = 1, \quad X \in \mathbb{R};\\
&|\nabla_{X,Y}\Psi(X, \xi(X))|^2 + 2\xi(X) = 3r, \quad X \in \mathbb{R}.
\end{aligned}$$

In the last relation (Bernoulli's equation), $r > 0$ is the problem's parameter referred to as Bernoulli's constant.

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$$\begin{aligned} &u'' + \omega(u) = 0, \quad u(0) = 0, \\ &u(h) = 1, \quad |u'(h)|^2 + 2h = 3r, \end{aligned} \tag{1}$$

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By $U(y; s)$, we denote a solution to $U'' + \omega(U) = 0$ satisfying

$$U(0; s) = 0, \quad U'(0; s) = s, \quad \text{where } s \in \mathbb{R}.$$

We parameterize solutions by

$$s \geq s_0 = \sqrt{2 \max_{0 \leq \tau \leq 1} \Omega(\tau)},$$

$$\Omega(\tau) = \int_0^\tau \omega(t)\,\mathrm{d}t. \tag{2}$$

If $h(s) > 0$ is found for some s so that

$$\begin{aligned} &U(h(s); s) = 1 \\ &s^2 - 2\,\Omega(U(h(s); s)) + 2h(s) = 3r, \end{aligned} \tag{3}$$

then the pair $(U(Y; s),\ h(s))$ is a stream solution. Here we used

$$(U')^2 + 2\,\Omega(U) = s^2$$

which follows from the first equation in (1).

To describe solutions we introduce $\tau_\pm(s)$ and $y_\pm(s)$: $(y_-(s),\, y_+(s))$ is the maximal interval, where $U(Y;s)$ increases strictly monotonically, and $\tau_+(s)$ $[\tau_-(s)]$ is the supremum [infimum, respectively] of $U(Y;s)$ on this interval.
One can show that $\tau_+(s)$ $[\tau_-(s)]$ is the smallest positive root [the largest negative root, respectively] of the equation $2\,\Omega(\tau) = s^2$ with $s \geq s_0$. If there is no finite positive [negative] root, then $\tau_+(s) = +\infty$ $[\tau_-(s) = -\infty$, respectively]. Notice that $\tau_+(s) \geq 1$ if and only if $s \geq s_0$. Furthermore

$$y_\pm(s) = \int_0^{\tau_\pm(s)} \frac{\mathrm{d}\tau}{\sqrt{s^2 - 2\Omega(\tau)}} \quad \text{for } s \geq s_0.$$

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- If both $y_+(s)$ and $y_-(s)$ are finite, then $U(Y;s)$ is harmonic-like; it attains one of its minima at $Y = y_-(s)$ and one of its maxima at $Y = y_+(s)$. Moreover, $U(Y;s)$ increases strictly monotonically from $\tau_-(s)$ to $\tau_+(s)$ on $[y_-(s), y_+(s)]$.

Furthermore, for $s > s_0$

$$h(s) = \int_0^1 \frac{\mathrm{d}\tau}{\sqrt{s^2 - 2\Omega(\tau)}} \tag{4}$$

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is the smallest positive value of the depth such that $U(h(s); s) = 1$. For $s \geq s_0$ we denote by $h_j^{(+)}(s)$, $j = 0, 1, \ldots$, the whole sequence (possibly finite) of depth values, which satisfy the equation $U(h(s); s) = 1$ for this s. These functions with even and odd numbers have the following form:

$$\begin{aligned} h_{2k}^{(+)}(s) &= h(s) + 2k\,[y_+(s) - y_-(s)]\,, \quad j = 2k; \\ h_{2k+1}^{(+)}(s) &= h(s) + 2\,[y_+(s) - h(s)] \\ &+2k\,[y_+(s) - y_-(s)]\,, \qquad j = 2k+1. \end{aligned}$$

Here, $k = 0, 1, \ldots$. If both $y_+(s)$ and $y_-(s)$ are finite, then the formulae give finite values for all $k = 0, 1, \ldots$.

The equation $U(h(s);s)=1$ has also the following sequence of solutions:

$$h_j^{(-)}(s)=h_j^{(+)}(s)+2y_-(-s)=h_j^{(+)}(s)-2y_-(s),$$
$$j=0,1,\ldots,$$

for $s>s_0$ and $U'(0)=-s$. +pause The number of finite elements depends on whether $y_+(s)$ and $y_-(s)$ are finite or not.

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If the depth is equal to $h_j^{(+)}$, then the number of layers with alternating directions of flow is equal to $j+1$, whereas the direction of flow in the near-bottom layer is positive. If the depth is equal to $h_j^{(-)}$, then there are $j+2$ layers with alternating directions of flow and the near-bottom layer is negative.
Finally, s is found from Bernoulli's equation

$$\mathcal{R}_j^{(\pm)}(s)=r$$
$$\mathcal{R}_j^{(\pm)}(s)=\frac{1}{3}\left[s^2-2\,\Omega(1)+2\,h_j^{(\pm)}(s)\right],\ j=0,1,\dots$$

Let us turn to r_c. Since $\mathcal{R}_j^{(+)}(s)$ and $\mathcal{R}_j^{(-)}(s)$ are increasing for every $s > s_0$, we have that the graph of $\mathcal{R}_0^{(+)}(s)$ lies below those corresponding to other functions $\mathcal{R}_j^{(\pm)}(s)$. Therefore

$$r_c = \min_{s \geq s_0} \mathcal{R}_0^{(+)}(s)\,. \tag{5}$$

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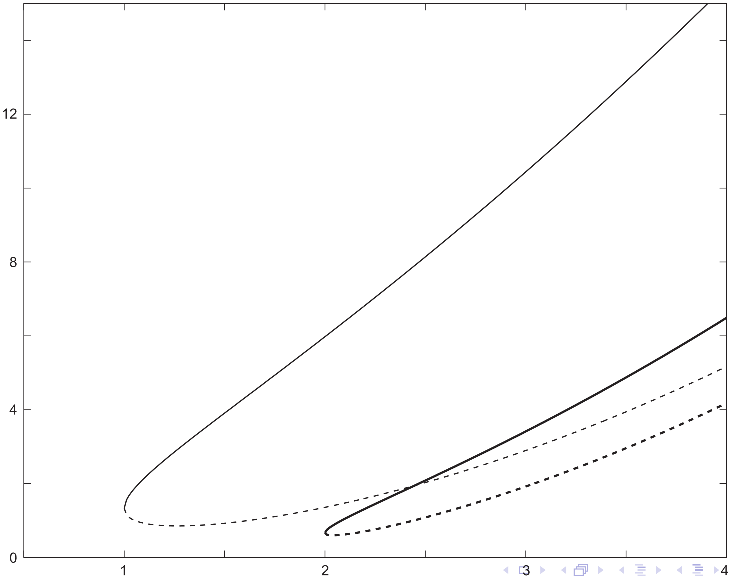
$$r_0 = \lim_{s \to s_0+0} \mathcal{R}_0^{(+)}(s) \tag{6}$$

is finite, then among the flows corresponding to every $r > r_0$ there are flows with counter-currents.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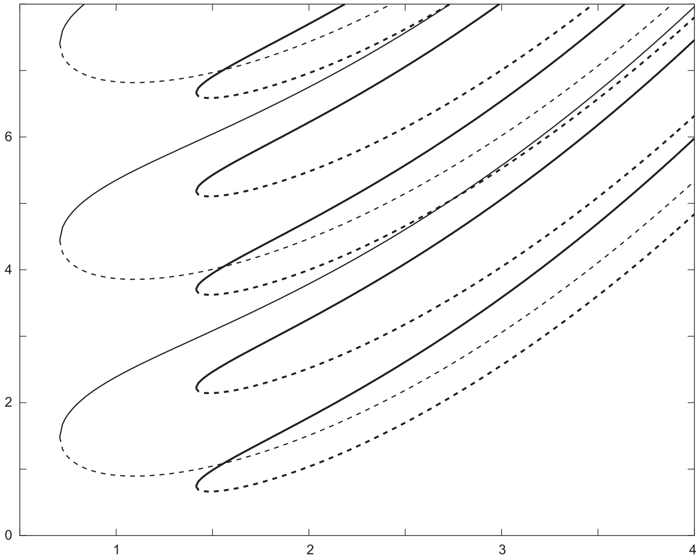
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(I) $r > r_c$, for every such r at least two roots exist and we choose one of them, say, s_* so that $s_* \neq s_0$, which is equivalent to $U'(h; s_*) \neq 0$.
In the irrotational case the *dispersion equation* can be written as

$$\tau \coth h\tau = h^2. \tag{7}$$

Here $h > 1$ is defined by r and is equal to the non-dimensional depth of the subcritical uniform stream. It is clear that (7) has only one positive root which is equal to the wavenumber $2\pi/\Lambda_0$ corresponding to the bifurcation wavelength.

In the rotational case, the chosen stream solution $(U(\cdot; s_*),\ h(s_*))$ yields the following dispersion equation:

$$\sigma(\tau) = 0, \quad \sigma(\tau) = \kappa\,\gamma'(h,\tau) - \kappa^{-1} + \omega(1), \tag{8}$$

where $\kappa = U'(h)$ and γ satisfies

$$-\gamma'' + [\tau^2 - \omega'(U)]\,\gamma = 0$$
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Note that h depends on r through the root s_* of Bernoulli's equation and $\kappa = \pm\sqrt{3r - 2h}$ when $\pm\,U'(h) > 0$. If τ^2 is not a Dirichlet eigenvalue of the operator $\mathrm{d}^2/\mathrm{d}Y^2 + \omega'(U)$, then problem (9) has a unique solution. Therefore, $\gamma(Y,\tau)$ and $\sigma(\tau)$ are defined for all such values of τ; moreover, they are smooth and even functions of τ.

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If ω vanishes identically, then (8) coincides with (7). Indeed, we have that $\kappa = h^{-1}$ in this case, whereas $U = Y/h$, and so $\gamma = \sinh Y\tau / \sinh h\tau$ is the solution of (9).

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Now we turn to the second assumption

(II) the dispersion equation (8) has at least one positive root, say, τ_0 such that: (a) $\sigma'(\tau_0) \neq 0$, which means that this root is simple; (b) none of τ_0/k ($k = 1, 2, \dots$) is a root of (8).

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Now we are in a position to formulate the following.

Definition of a bifurcation wavelength. Let assumptions (I) and (II) hold, and let τ_0 be a root of (8). Then we put $\Lambda_0 = 2\pi/\tau_0$.

If equation (8) has more than one root satisfying conditions (a) and (b) of assumption (II), then each of these roots defines a wavelength of linear vortical waves and all these waves exist on the free surface of one and the same shear flow.

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Our last assumption says that

(III) if the boundary value problem

$$\varphi_{XX} + \varphi_{YY} + \omega'(U)\,\varphi = 0 \quad \text{in } \mathbb{R} \times (0, h)$$
$$\varphi(X, 0) = 0, \quad \varphi(X, h) = 0,$$

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In particular, this implies that zero is not a Dirichlet eigenvalue of the operator $\mathrm{d}^2/\mathrm{d}Y^2 + \omega'(U)$ considered on $(0, h)$, and so the function $\sigma(\tau)$ is defined for τ belonging to a neighbourhood of zero.

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Let $\omega \in C^{2,\alpha}(\mathbb{R})$, $\alpha \in (0,1)$.

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(i) $\lambda(t) \not\equiv 0$, $\lambda(t) \to 0$ as $|t| \to 0$, and

$$\zeta(x,t) = t \cos \frac{2\pi x}{\Lambda_0} + \zeta_*(x,t), \qquad (10)$$

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where $\|\zeta_*(\cdot, t)\|_{\Pi^{2,\alpha}_{\Lambda_0}} = o(t)$ as $|t| \to 0$.

(ii) If the upper boundary of $\mathcal{D}$ has the form

$$\xi(X,\, t) = \frac{h}{1+\lambda(t)} + \zeta\left(\frac{X}{1+\lambda(t)}, t\right), \tag{11}$$

that is, $\xi(X,\, t)$ a Λ-periodic function of X with $\Lambda = \Lambda_0[1+\lambda(t)]$, then

$$\Psi(X,Y,t) = U\left(Y \frac{h}{\xi(X,t)}\right) + \Psi_*(X,Y,t), \tag{12}$$

where $\|\Psi_*(\cdot, t)\|_{\Pi^{2,\alpha}_{\Lambda}(\bar{\mathcal{D}})} = O(t)$ as $t \to 0$.

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2). *Let* τ_*^2 *be a non-zero Dirichlet eigenvalue of* $\mathrm{d}^2/\mathrm{d}Y^2 + \omega'(U)$ *considered on* $(0, h)$*, then*

$$\sigma(\tau) = \frac{-\kappa[\gamma_*'(h)]^2}{2\tau_*(\tau - \tau_*)} + O(1) \quad \textit{as } \tau \to \tau_*. \tag{14}$$

Here $\gamma_*(Y)$ *is a corresponding eigenfunction normalized in* $L^2(0, h)$*; moreover* $\gamma_*'(h) \neq 0$*.*

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(i) *If the operator* $\mathrm{d}^2/\mathrm{d}Y^2 + \omega'(U)$ *considered on* $(0, h)$ *has no Dirichlet eigenvalues, then the following equality holds:*

$$\sigma(0) = -\frac{2}{3\kappa}\left[\frac{\mathrm{d}\mathcal{R}}{\mathrm{d}s}(s_*) \Big/ \frac{\mathrm{d}h}{\mathrm{d}s}(s_*)\right]. \tag{15}$$

Moreover, if

$$\frac{\mathrm{d}\mathcal{R}}{\mathrm{d}s}(s_*) \Big/ \frac{\mathrm{d}h}{\mathrm{d}s}(s_*) > 0, \tag{16}$$

then equation (8) *has at least one positive root.*

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then equation (8) *has at least one positive root.*
(ii) *Let zero be not an eigenvalue of the operator* $\mathrm{d}^2/\mathrm{d}Y^2 + \omega'(U)$ *considered on* $(0, h)$. *If this operator has exactly* k *positive Dirichlet eigenvalues, then* (8) *has at least* k *positive roots. Moreover, if inequality* (16) *holds, then* (8) *has at least* $k + 1$ *positive roots.*

A stream solution (U, h) describes a unidirectional flow $(U'(Y) > 0$ for all $Y \in [0, h])$ only when this solution depends on $s > s_0$ that is a root of equation $\mathcal{R}_0^{(+)}(s) = r$.

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Second, the corresponding operator $\mathrm{d}^2/\mathrm{d}Y^2 + \omega'(U)$ considered on $(0, h)$ has the empty Dirichlet spectrum. Hence the solution $\gamma(Y; \tau)$ of problem (9) (and therefore σ) is defined for all $\tau \in \mathbb{R}$ and is a smooth function of both variables.

Proposition 3.5. *Let $s > s_0$ be a root of the equation $\mathcal{R}_0^{(+)}(s) = r$ with some $r > r_c$.*

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Proposition 3.5. *Let $s > s_0$ be a root of the equation $\mathcal{R}_0^{(+)}(s) = r$ with some $r > r_c$.If the function σ is defined by the stream solution (U, h) that corresponds to s, then the following assertions hold.*

A stream solution (U, h) describes a unidirectional flow ($U'(Y) > 0$ for all $Y \in [0, h]$) only when this solution depends on $s > s_0$ that is a root of equation $\mathcal{R}_0^{(+)}(s) = r$.For $\mathcal{R}_0^{(+)}$ we have

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Proposition 3.5. *Let $s > s_0$ be a root of the equation $\mathcal{R}_0^{(+)}(s) = r$ with some $r > r_c$.If the function σ is defined by the stream solution (U, h) that corresponds to s, then the following assertions hold.*

(i) *For any $s \in (s_0, s_c)$ equation (8) has a positive solution.*

A stream solution (U, h) describes a unidirectional flow ($U'(Y) > 0$ for all $Y \in [0, h]$) only when this solution depends on $s > s_0$ that is a root of equation $\mathcal{R}_0^{(+)}(s) = r$.For $\mathcal{R}_0^{(+)}$ we have

$$\pm \frac{\mathrm{d}\mathcal{R}_0^{(+)}}{\mathrm{d}s}(s) > 0 \quad \text{provided} \quad \pm(s - s_c) > 0\,. \tag{17}$$

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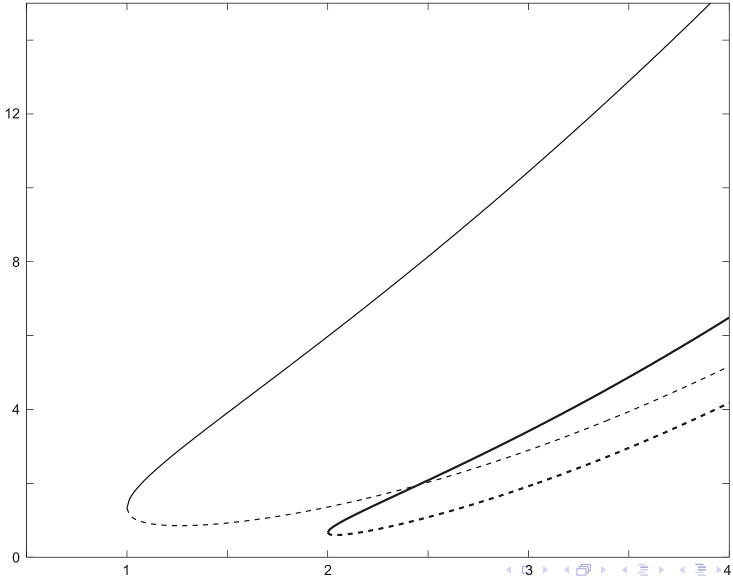
(i) *For any $s \in (s_0, s_c)$ equation (8) has a positive solution.*
(ii) *If $s > s_c$, then equation (8) has no positive solutions.*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Let $\omega = b$ be a positive constant, then

$$h_0^{(+)}(s) = \frac{s - \sqrt{s^2 - 2b}}{b}$$
$$h_1^{(+)}(s) = \frac{s + \sqrt{s^2 - 2b}}{b} \qquad (18)$$

are the only non-vanishing depth functions defined for $s > s_0$ and such that $h_0^{(+)}(s) > h_1^{(+)}(s)$.+pause Hence

$$\mathcal{R}^{(+)}_{\frac{1}{2}\mp\frac{1}{2}}(s) = \frac{1}{3}\left[s^2 - 2b + 2\frac{s \mp \sqrt{s^2 - 2b}}{b}\right]. \qquad (19)$$

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They have the following properties: $\mathcal{R}_0^{(+)}(s) < \mathcal{R}_1^{(+)}(s)$ for $s > s_0$, while

$$r_0 = \mathcal{R}_0^{(+)}(s_0) = \mathcal{R}_1^{(+)}(s_0) = \frac{2}{3}\sqrt{\frac{2}{b}} < +\infty.$$

We have that

$$\frac{\mathrm{d}\mathcal{R}^{(+)}_{\frac{1}{2}\mp\frac{1}{2}}}{\mathrm{d}s}(s) \to \mp\infty \quad \text{as } s \to s_0 + 0$$
$$\frac{\mathrm{d}\mathcal{R}^{(+)}_{1}}{\mathrm{d}s}(s) > 0 \quad \text{for } s > s_0. \tag{20}$$

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The first of these relations shows that the graphs of $\mathcal{R}^{(+)}_0$ and $\mathcal{R}^{(+)}_1$ have the common vertical tangent at the point (s_0, r_0), while the latter inequality implies that the function $\mathcal{R}^{(+)}_1$ increases strictly monotonically, and so its graph of lies strictly above the level r_0.

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The first of these relations shows that the graphs of $\mathcal{R}_0^{(+)}$ and $\mathcal{R}_1^{(+)}$ have the common vertical tangent at the point (s_0, r_0), while the latter inequality implies that the function $\mathcal{R}_1^{(+)}$ increases strictly monotonically, and so its graph of lies strictly above the level r_0.Finally, the single minimum $r_c < r_0$ of $\mathcal{R}_0^{(+)}$ is attained at $s_c > s_0$ satisfying the following equation:

$$(s_c b)^2 - 2b^3 = \left(\frac{s_c b}{1 + s_c b}\right)^2. \tag{21}$$

In order to write down the dispersion equation we find from problem (9) that

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Here and below $h = h(s)$ is either $h_0^{(+)}(s)$ or $h_1^{(+)}(s)$. We substitute $\kappa = -(bh - s)$ and γ into (8) and obtain

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This dispersion equation coincides with (7) when $b = 0$; indeed, we have that $s = h = h_0^{(+)}$ in view of the first formula (18) [the second formula (18) is meaningless for $b = 0$].

First, let $r > r_0$, then the function $\mathcal{R}_1^{(+)}$ increases monotonically. Therefore, the inequality

$$\frac{\mathrm{d}h_1^{(+)}}{\mathrm{d}s}(s) = \frac{1}{b}\left(1 + \frac{s}{\sqrt{s^2 - 2b}}\right) > 0$$

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$$\frac{\mathrm{d}h_0^{(+)}}{\mathrm{d}s}(s) = \frac{1}{b}\left(1 - \frac{s}{\sqrt{s^2 - 2b}}\right) < 0 \quad \text{for all } s > s_0,$$

and the following inequality holds

$$\frac{\mathrm{d}\mathcal{R}_0^{(+)}}{\mathrm{d}s}(s) = \frac{2}{3}\left[s + \frac{1}{b}\left(1 - \frac{s}{\sqrt{s^2 - 2b}}\right)\right] < 0$$

for $s \in (s_0, s_c)$.

Proposition 5.1. *Let $\omega = b$, where b is a positive constant, then the following assertions hold.*

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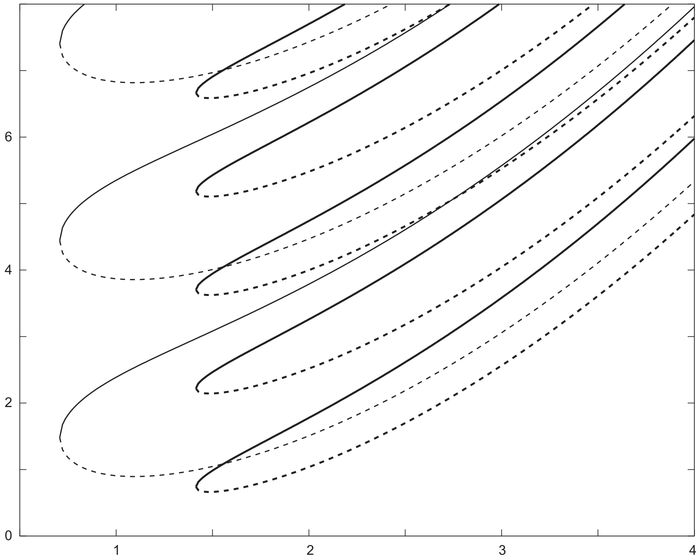
(i) *For every $r > r_0$ the equation $\mathcal{R}_1^{(+)}(s) = r$ for $\mathcal{R}_1^{(+)}$] has the unique root $s_*^{(1)} > s_0$. In the flow of the constant depth $h = h_1^{(+)}\left(s_*^{(1)}\right)$, the velocity field is described by the stream function. The dispersion equation with $s = s_*^{(1)}$ and $h = h_1^{(+)}\left(s_*^{(1)}\right)$ has only one positive simple root.*+pause

(ii) *For every $r \in (r_c, r_0)$ the equation $\mathcal{R}_0^{(+)}(s) = r$ has the unique root $s_*^{(0)} \in (s_0, s_c)$. In the flow of the constant depth $h = h_0^{(+)}\left(s_*^{(0)}\right)$, the velocity field is described by the stream function. The dispersion equation with $s = s_*^{(0)}$ and $h = h_0^{(+)}\left(s_*^{(0)}\right)$ has only one positive simple root.*

Corollary 5.2. *Let $\omega = b$, where b is a positive constant, then for every $r > r_c$, $r \neq r_0$, there exists a family of Stokes waves with wavelengths belonging to a neighbourhood of Λ_0 such that $2\pi/\Lambda_0$ is the only positive root of the dispersion equation depending on r. These small-amplitude waves perturb each of the flows whose constant depths.*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$$h_j^{(+)}(s) = \frac{(-1)^j}{\sqrt{b}} \arcsin \frac{\sqrt{b}}{s} + j\frac{\pi}{\sqrt{b}} \tag{23}$$

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give the depths of flows with horizontal free surfaces. Therefore,

$$\mathcal{R}_j^{(\pm)}(s) = \frac{1}{3}\Big[s^2 - b + \frac{2(-1)^j}{\sqrt{b}} \arcsin \frac{\sqrt{b}}{s} + \frac{2\pi}{\sqrt{b}}\Big(j + \frac{1}{2} \mp \frac{1}{2}\Big)\Big], \; j = 0, 1, \ldots . \tag{24}$$

In particular, we have that

$$\mathcal{R}_0^{(+)}(s) = \frac{1}{3}\left(s^2 - b + \frac{2}{\sqrt{b}}\arcsin\frac{\sqrt{b}}{s}\right)$$

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Hence

$$r_0 = \mathcal{R}_0^{(+)}(s_0) = \mathcal{R}_1^{(+)}(s_0) = \frac{\pi}{3\sqrt{b}},$$

whereas r_c - the only minimum of $\mathcal{R}_0^{(+)}(s)$ - is attained at

$$s_c = \sqrt{\frac{b}{2} + \sqrt{\left(\frac{b}{2}\right)^2 + 1}}. \tag{25}$$

The left-hand side of the dispersion equation involves γ that solves the following problem:

$$-\gamma'' + (\tau^2 - b)\,\gamma = 0, \quad \gamma(0) = 0, \;\; \gamma(h) = 1, \qquad (26)$$

where $h = h_j^{(\pm)}(s)$, $j = 1, 2, \ldots$.

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This condition is obviously fulfilled when $\tau^2 \geq b$, in which case $\gamma(Y, \tau)$ is equal to

$$\frac{\sinh \sqrt{\tau^2 - b}\, Y}{\sinh \sqrt{\tau^2 - b}\, h} \quad \text{when } \tau^2 > b,$$

and so γ (and σ as well) is a continuous function of τ when $\tau^2 \geq b$.

We get that the sequence of dispersion equations has the form for $\tau^2 \geq b$

$$s^2\sqrt{\tau^2 - b}\cos^2\left(\sqrt{b}\, h_j^{(\pm)}(s)\right)\coth\left(\sqrt{\tau^2 - b} h_j^{(\pm)}(s)\right)$$
$$-1 \pm bs\, \cos\left(\sqrt{b}\, h_j^{(\pm)}(s)\right) = 0, j = 0, 1, \dots,$$

whereas for $\tau^2 = b$ the left-hand side must be understood as the corresponding limit.

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If $\tau^2 < b$, then

$$\gamma(Y,\tau) = \frac{\sin\sqrt{b-\tau^2}\,Y}{\sin\sqrt{b-\tau^2}\,h}$$

and we get the sequence of dispersion equations for this case:

$$s^2\sqrt{b-\tau^2}\,\cos^2\left(\sqrt{b}\,h_j^{(\pm)}(s)\right)\cot\left(\sqrt{b-\tau^2}\,h_j^{(\pm)}(s)\right)$$
$$-1 \pm bs\,\cos\left(\sqrt{b}\,h_j^{(\pm)}(s)\right) = 0, j = 0, 1, \ldots$$

The case $s = s_0$

THANK YOU