Mechanical Balance Laws in the Boussinesq Scaling: Theory and Applications

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Joint work with:

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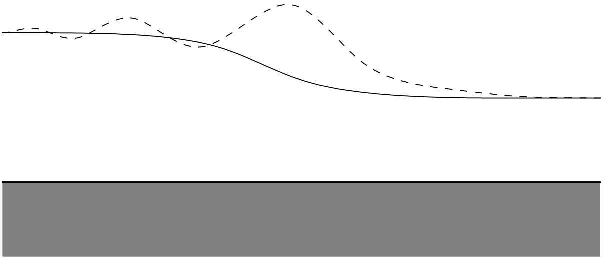
The river bore

A bore is a transition between two uniform flows in a river, usually caused by tidal forces.

Schematic of a possible two-dimensional bore profile:



Turbulent bore

China Daily, 9/19/2005:

Huge tide expected in Qiantang this month

Hangzhou: A powerful, massive tide is expected to sweep over the Qiantang River in Hangzhou, capital city of East China's Zhejiang Province, later this month, posing a potential danger to people outside the designated tide-watching areas, experts said yesterday. The tidal waves are likely to reach a height of 2.5 metres, much higher than that of past years, said Bao Yuepeng, director ...

Shallow-water approximation: $\frac{h_0}{\lambda} \ll 1$

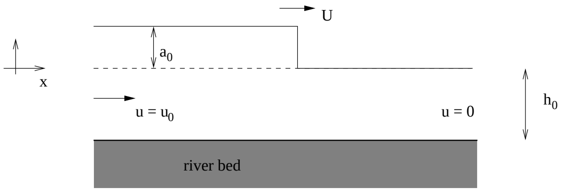
Assumptions:

- $p = (\eta - z)g$
- $u = u(x,t)$

Shallow-water equations:

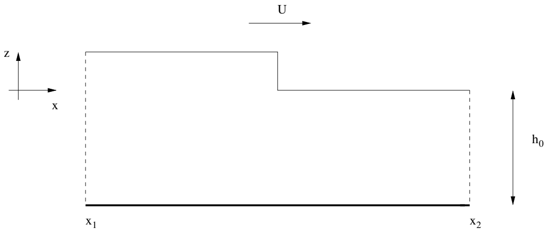
$$\begin{aligned} \eta_t + h_0 u_x + (\eta u)_x &= 0 \\ u_t + g\eta_x + u u_x &= 0 \end{aligned}$$

An exact weak solution:



Shallow-water approximation

Control volume:



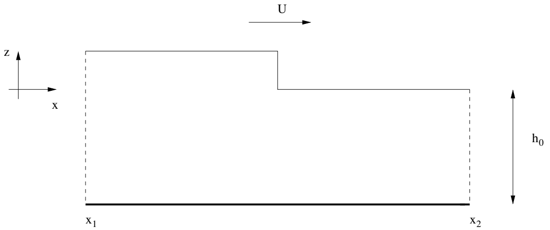
Let $h(x,t) = h_0 + \eta(x,t)$

Conservation of mass:

$$\frac{d}{dt}\int_{x_1}^{x_2} \rho h(x,t)\,dx = \rho u(x_1,t)h(x_1,t) - \rho u(x_2,t)h(x_2,t)$$

Shallow-water approximation

Control volume:

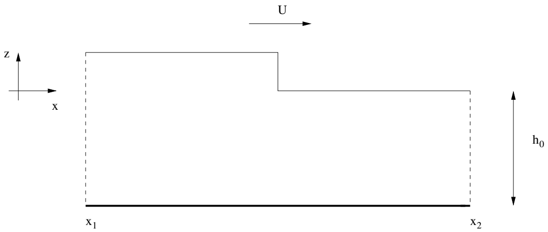


Conservation of momentum:

$$\frac{d}{dt}\int_{x_1}^{x_2} \rho u(x,t)h(x,t)\,dx = \rho u^2(x_1,t)h(x_1,t) - \rho u^2(x_2,t)h(x_2,t) + \frac{\rho}{2}gh^2(x_1,t)h(x_1,t) - \frac{\rho}{2}gh^2(x_2,t)h(x_2,t)$$

Shallow-water approximation

Control volume:

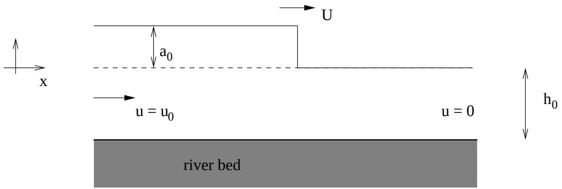


Jump conditions:

$$-U\,[h]_{x_1}^{x_2} + [uh]_{x_1}^{x_2} = 0$$

$$-U\,[uh]_{x_1}^{x_2} + \left[u^2 h + \frac{1}{2} g h^2\right]_{x_1}^{x_2} = 0$$

Shallow-water approximation



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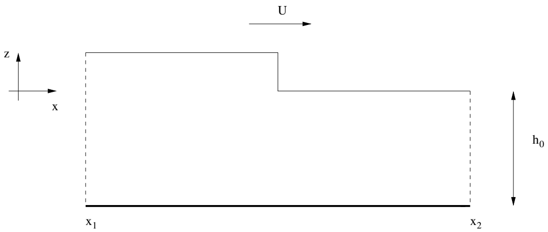
$\Longrightarrow$ U and u_0 are given in terms of a_0 and h_0

Shallow-water approximation: Energy loss



Mechanical energy is given by

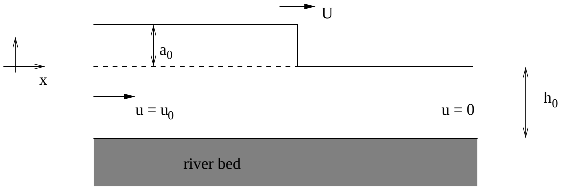
$$\mathcal{E}_{sw} = \frac{\rho}{2} \int_{x_1}^{x_2} \left\{ u^2(x,t)h(x,t) + gh^2(x,t) \right\} dx$$

Energy flux is given by

$$(q_E)_1 = \frac{\rho}{2} u^3(x_1,t)h(x_1,t) + \rho g u(x_1,t)h^2(x_1,t)$$

$$(q_E)_2 = \frac{\rho}{2} u^3(x_2,t)h(x_2,t) + \rho g u(x_2,t)h^2(x_2,t)$$

Shallow-water approximation: Energy loss

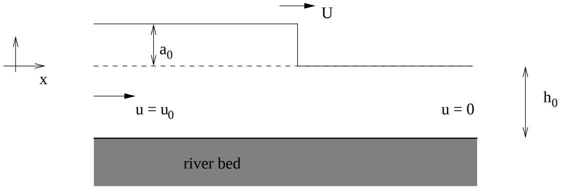


Lord Rayleigh, 1876

Loss of energy:

$$-\frac{d\mathcal{E}_{sw}}{dt} + \left[(q_E)_1 - (q_E)_2\right] = \frac{a_0^3}{4}\rho\sqrt{\frac{1}{2}\,g^3\,\left(\frac{1}{h_0} + \frac{1}{a_0+h_0}\right)}$$

Bore types



Bore types:

- $\frac{a_0}{h_0} < 0.28 \Longrightarrow$ undular bore
- $0.28 < \frac{a_0}{h_0} < 0.75 \Longrightarrow$ undular bore, some waves are breaking
- $\frac{a_0}{h_0} > 0.75 \Longrightarrow$ turbulent bore

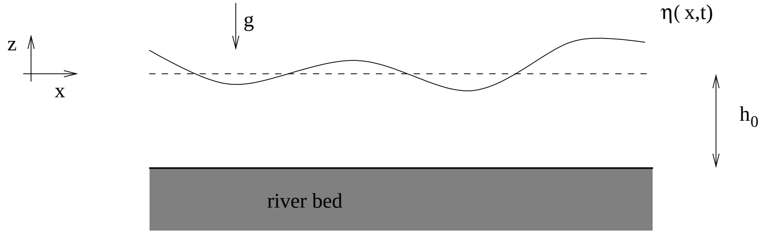
Folklore: Energy loss in undular bore is due to oscillations
Energy loss in turbulent bore is due to turbulent dissipation

Surface gravity waves

Surface gravity waves

Assumptions:

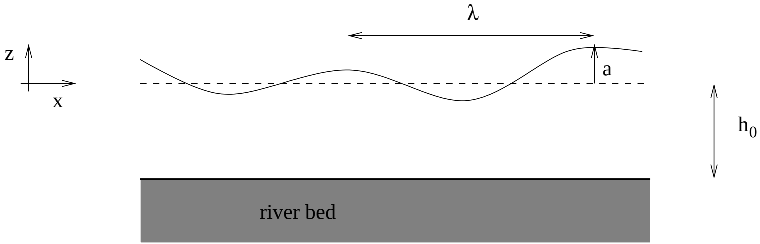
- incompressible
- inviscid
- two-dimensional
- irrotational
- unit density



Surface gravity waves

Long wavelength: $\frac{h_0}{\lambda} \ll 1$

Small amplitude: $\frac{a}{h_0} \ll 1$



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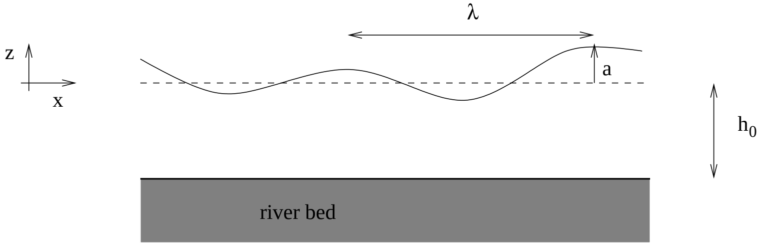
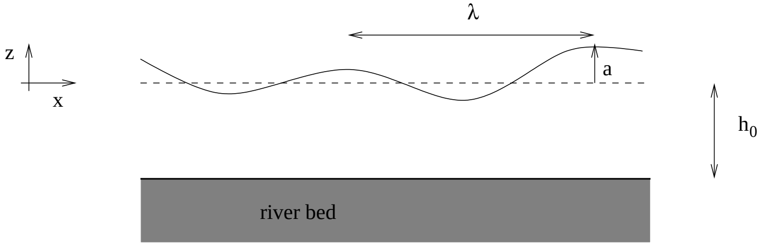


Surface gravity waves

Long wavelength: $\frac{h_0}{\lambda} \ll 1$ $\Longrightarrow$ **Shallow water**

Small amplitude: $\frac{a}{h_0} \ll 1$



Shallow-water equations:

$$\begin{aligned} \eta_t + h_0 u_x + (\eta u)_x &= 0 \\ u_t + g\eta_x + u u_x &= 0 \end{aligned}$$

Assumptions:

- $p = (\eta - z)g$ (hydrostatic)
- $u = u(x,t)$ (no vertical acceleration)

Associated balance laws:

$$\frac{\partial}{\partial t} M + \frac{\partial}{\partial x} q_M = 0 \qquad \text{(mass balance)}$$

$$\frac{\partial}{\partial t} I + \frac{\partial}{\partial x} q_I = 0 \qquad \text{(momentum balance)}$$

$$\begin{aligned} M &= h_0 + \eta \\ q_M &= h_0 u + \eta u \\ I &= h_0 u + \eta u \\ q_I &= (h_0 + \eta) u^2 + \tfrac{1}{2} g (h_0 + \eta)^2 \end{aligned}$$

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$$\frac{\partial}{\partial t}E + \frac{\partial}{\partial x}q_E = 0 \qquad \text{(energy balance)}$$

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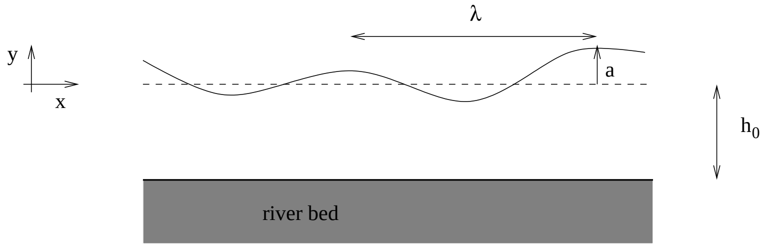
$$\frac{\partial}{\partial t} E + \frac{\partial}{\partial x} q_E = 0 \qquad \text{(energy balance)}$$

$$\begin{aligned} E &= \tfrac{1}{2}(h_0 + \eta) u^2 + \tfrac{1}{2} g (h_0 + \eta)^2 \\ q_E &= \tfrac{1}{2}(h_0 + \eta) u^3 + g (h_0 + \eta)^2 u \end{aligned}$$

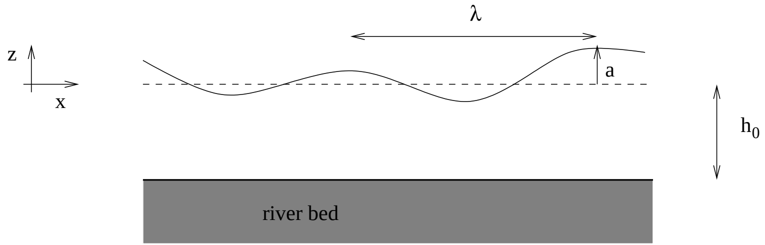
Long wavelength: $\frac{h_0}{\lambda} \ll 1$

Small amplitude: $\frac{a}{h_0} \ll 1$



Long wavelength: $\frac{h_0}{\lambda} \ll 1$

Small amplitude: $\frac{a}{h_0} \ll 1$ $\Longrightarrow$ **Airy theory**



Dispersion relation:

$$\begin{aligned}\eta &= A\cos(kx - \omega t)\\ \phi &= Z(z)\sin(kx - \omega t)\end{aligned}$$

$$\Longrightarrow \quad \omega^2 = gk\tanh(h_0 k)$$

Linear phase velocity:

$$c(k) = \frac{\omega}{k} = \sqrt{\frac{g}{k}\tanh(h_0 k)},$$

for waves propagating to the right.

Group velocity:

$$c_g = \frac{d\omega}{dk} = \frac{1}{2}c(k)\left(1 + \frac{2kh_0}{\sinh 2kh_0}\right)$$

Flow field:

$$u = \quad A\omega \frac{\cosh k(z+h_0)}{\sinh kh_0} \cos(kx - \omega t)$$

$$v = \quad A\omega \frac{\sinh k(z+h_0)}{\sinh kh_0} \sin(kx - \omega t)$$

$$p = \quad gA \frac{\cosh k(z+h_0)}{\cosh kh_0} \cos(kx - \omega t) - gz$$

Energy density of (linear) progressive wave:

$$E = \frac{1}{2} gA^2$$

Energy flux:

$$q_E = \left[\frac{1}{2} gA^2\right] \times \left[\frac{c}{2}\left(1 + \frac{2kh_0}{\sinh 2kh_0}\right)\right] = E \times c_g$$

Boussinesq models

Long wavelength: $\frac{h_0}{\lambda} \ll 1$

Small amplitude: $\frac{a}{h_0} \ll 1$

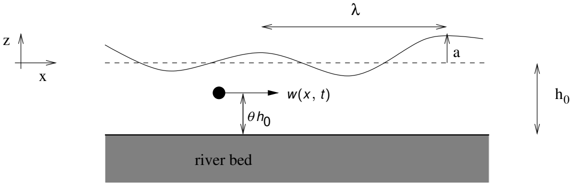
Boussinesq scaling:

$$\alpha = \frac{a}{h_0} \sim \frac{h_0^2}{\lambda^2} = \beta$$

General system:

$$\eta_t+h_0 w_x+(\eta w)_x+\frac{1}{2}\left(\theta^2-\frac{1}{3}\right)\lambda h_0^3 w_{xxx}-\frac{1}{2}\left(\theta^2-\frac{1}{3}\right)(1-\lambda)h_0^2\eta_{xxt}=0,$$

$$w_t+g\eta_x+ww_x+\frac{1}{2}\left(1-\theta^2\right)\mu g h_0^2\eta_{xxx}-\frac{1}{2}\left(1-\theta^2\right)(1-\mu)h_0^2 w_{xxt}=0.$$



Horizontal velocity is represented at θh_0, where $0 \leq \theta \leq 1$
Additional modeling parameters: λ and μ

Special cases

Classical Boussinesq system with $\theta^2 = \frac{1}{3}$ and $\mu = 0$,

$$\eta_t + h_0 w_x + (\eta w)_x = 0$$
$$w_t + g\eta_x + ww_x - \frac{h_0^2}{3} w_{xxt} = 0$$

Kaup System with $\theta^2 = 1,\ \lambda = 1$,

$$\eta_t + h_0 w_x + (\eta w)_x - \frac{h_0^3}{3} w_{xxx} = 0$$
$$w_t + g\eta_x + ww_x = 0$$

KdV-KdV system with $\theta^2 = \frac{2}{3}$, $\lambda = 1$ and $\mu = 1$

$$\eta_t + h_0 w_x + (\eta w)_x + \frac{1}{6} h_0^3 w_{xxx} = 0$$
$$w_t + g\eta_x + ww_x + \frac{1}{6} g h_0^2 \eta_{xxx} = 0$$

Derivation of evolution equations

Non-dimensional variables

$$\tilde{x} = \frac{x}{\ell}, \quad \tilde{z} = \frac{z + h_0}{h_0}, \quad \tilde{\eta} = \frac{\eta}{a}, \quad \tilde{t} = \frac{c_0 t}{\ell}, \quad \tilde{\phi} = \frac{c_0}{ga\ell}\phi,$$

Velocity potential

$$\tilde{\phi} \quad = \quad \tilde{f} - \frac{\tilde{z}^2}{2}\tilde{f}_{\tilde{x}\tilde{x}}\beta + \frac{\tilde{z}^4}{24}\tilde{f}_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}}\beta^2 + \mathcal{O}(\beta^3)$$

Substitute $\tilde{\phi}$ into the free-surface boundary conditions

$$\tilde{\eta} + \tilde{f}_{\tilde{t}} - \frac{\beta}{2}\tilde{f}_{\tilde{x}\tilde{x}\tilde{t}} + \frac{\alpha}{2}\tilde{f}_{\tilde{x}}^2 = \mathcal{O}(\alpha\beta, \beta^2),$$

Evolution equations:

$$\tilde{\eta}_{\tilde{t}} + \tilde{v}_{\tilde{x}} + \alpha(\tilde{\eta}\tilde{v})_{\tilde{x}} - \frac{1}{6}\beta\tilde{v}_{\tilde{x}\tilde{x}\tilde{x}} = \mathcal{O}(\alpha^2, \alpha\beta, \beta^2)$$
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where $\tilde{v} = f_{\tilde{x}}$, the non-dimensional horizontal velocity at the bottom.
Horizontal velocity at non-dimensional height θ is

$$\tilde{\phi}_{\tilde{x}}\Big|_{\tilde{z}=\theta} = \tilde{w} = \tilde{v} - \frac{\theta^2}{2}\tilde{v}_{\tilde{x}\tilde{x}}\beta + \mathcal{O}(\beta^2)$$

Velocity potential

$$\tilde{\phi} \quad = \quad \tilde{f} - \frac{\tilde{z}^2}{2}\tilde{f}_{\tilde{x}\tilde{x}}\beta + \frac{\tilde{z}^4}{24}\tilde{f}_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}}\beta^2 + \mathcal{O}(\beta^3)$$

Substitute $\tilde{\phi}$ into the free-surface boundary conditions

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New system:

$$\tilde{\eta}_{\tilde{t}} + \tilde{w}_{\tilde{x}} + \alpha(\tilde{\eta}\tilde{w})_{\tilde{x}} + \frac{1}{2}\left(\theta^2 - \frac{1}{3}\right)\beta\tilde{w}_{\tilde{x}\tilde{x}\tilde{x}} = \mathcal{O}(\alpha^2, \alpha\beta, \beta^2)$$
$$\tilde{\eta}_{\tilde{x}} + \tilde{w}_{\tilde{t}} + \alpha\tilde{w}\tilde{w}_{\tilde{x}} + \frac{1}{2}\beta\left(\theta^2 - 1\right)\tilde{w}_{\tilde{x}\tilde{x}\tilde{t}} = \mathcal{O}(\alpha^2, \alpha\beta, \beta^2)$$

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Note the first-order relations

$$\tilde{\eta}_{\tilde{t}} + \tilde{w}_{\tilde{x}} = \mathcal{O}(\alpha, \beta)$$
$$\tilde{w}_{\tilde{t}} + \tilde{\eta}_{\tilde{x}} = \mathcal{O}(\alpha, \beta)$$

Differentiate first one:

$$\tilde{\eta}_{\tilde{t}\tilde{x}\tilde{x}} + \tilde{w}_{\tilde{x}\tilde{x}\tilde{x}} = \mathcal{O}(\alpha, \beta)$$

New system:

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How to find associated densities and fluxes?

Idea:

$$\frac{\partial}{\partial \tilde{t}} \tilde{M} + \frac{\partial}{\partial \tilde{x}} \tilde{q}_M = \mathcal{O}(\alpha^2, \alpha\beta, \beta^2)$$

$$\frac{\partial}{\partial \tilde{t}} \tilde{I} + \frac{\partial}{\partial \tilde{x}} \tilde{q}_I = \mathcal{O}(\alpha^2, \alpha\beta, \beta^2)$$

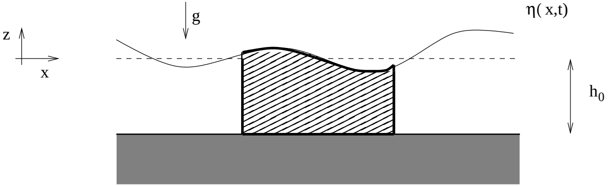
$$\frac{\partial}{\partial \tilde{t}} \tilde{E} + \frac{\partial}{\partial \tilde{x}} \tilde{q}_E = \mathcal{O}(\alpha^2, \alpha\beta, \beta^2)$$

Mass conservation

Consider the total mass in a control volume, given by

$$\mathcal{M} = \int_{x_1}^{x_2} \int_{-h_0}^{\eta} dzdx$$

$$= \int_{x_1}^{x_2} (h_0 + \eta)\, dx$$



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Mass flux is given by

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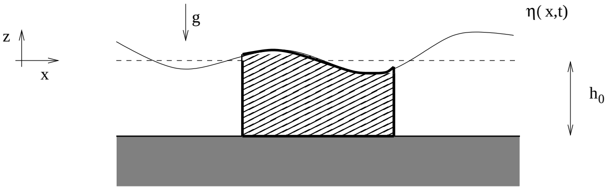

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Mass conservation:

$$\frac{d}{dt}\mathcal{M} = \left[q_m \right]_{x_2}^{x_1}$$

Mass conservation

Mass conservation:

$$\frac{d}{dt}\int_{x_1}^{x_2}\int_{-h_0}^{\eta} dzdx = \left[\int_{-h_0}^{\eta} \phi_x(x,z)\, dz\right]_{x_2}^{x_1}$$

In non-dimensional form:

$$\frac{d}{d\tilde{t}}\int_{x_1/\ell}^{x_2/\ell}\int_{0}^{1+\alpha\tilde{\eta}} d\tilde{z}d\tilde{x} = \alpha\left[\int_{0}^{1+\alpha\tilde{\eta}} \tilde{\phi}_{\tilde{x}}(\tilde{x},\tilde{z})\, d\tilde{z}\right]_{x_2/\ell}^{x_1/\ell}$$

Integrate with respect to $\tilde{z}$:

$$\begin{aligned}\frac{d}{d\tilde{t}}\int_{x_1/\ell}^{x_2/\ell}(1+\alpha\tilde{\eta})\, d\tilde{x} &= \alpha\left[\int_{0}^{1+\alpha\tilde{\eta}} \left\{\tilde{v} - \tfrac{\tilde{z}^2}{2}\beta\tilde{v}_{\tilde{x}\tilde{x}} + \mathcal{O}(\beta^2)\right\}\, d\tilde{z}\right]_{x_2/\ell}^{x_1/\ell}\\ &= \alpha\left[\tilde{v} + \alpha\tilde{v}\tilde{\eta} - \tfrac{\beta}{6}\tilde{v}_{\tilde{x}\tilde{x}} + \mathcal{O}(\alpha\beta,\beta^2)\right]_{x_2/\ell}^{x_1/\ell}\end{aligned}$$

Mass conservation

Take average over interval:

$$\frac{1}{x_2/\ell - x_1/\ell}\int_{x_1/\ell}^{x_2/\ell} \tilde{\eta}_{\tilde{t}} d\tilde{x} = \frac{1}{x_2/\ell - x_1/\ell}\left[\tilde{w} + \alpha\tilde{w}\tilde{\eta} - \frac{\beta}{2}(\theta^2 - \frac{1}{3})\tilde{w}_{\tilde{x}\tilde{x}}\right]_{x_2/\ell}^{x_1/\ell} + \mathcal{O}(\alpha\beta, \beta^2)$$

In the limit:

$$\tilde{\eta}_{\tilde{t}} + \tilde{w}_x + \alpha(\tilde{w}\tilde{\eta})_{\tilde{x}} - \frac{\beta}{2}(\theta^2 - \frac{1}{3})\tilde{w}_{\tilde{x}\tilde{x}\tilde{x}} = \mathcal{O}(\alpha\beta, \beta^2)$$

$\Longrightarrow$ mass density and flux are given in dimensional form by

$$M = h_0 + \eta,$$

and

$$q_M = h_0 w + \eta w + \frac{h_0^3}{2}\left(\theta^2 - \frac{1}{3}\right) w_{xx}.$$

Momentum conservation

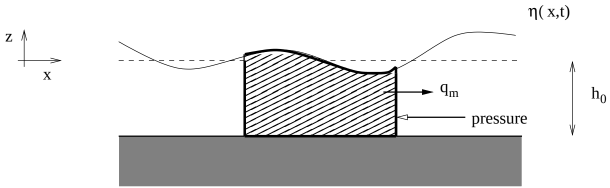
Pressure is given by

$$P = P_{atm} + g(\eta - z) + \frac{1}{2}\left((z + h_0)^2 - h_0^2\right) w_{xt}$$

Change of momentum I is equal to

- net influx of momentum through the boundaries
- plus net force on the boundary

$$\frac{d}{dt}\mathcal{I} = \left[q_I + \text{pressure force} \right]_{x_2}^{x_1},$$



Momentum conservation

Then the momentum balance is

$$\frac{\partial}{\partial \tilde{t}}\tilde{I} + \frac{\partial}{\partial \tilde{x}}\tilde{q}_I = \mathcal{O}(\alpha^2, \alpha\beta, \beta^2).$$

Dimensional forms of momentum density and momentum flux are

$$I = (h_0 + \eta)w + \frac{1}{2}\left(\theta^2 - \frac{1}{3}\right) h_0^3 w_{xx}$$

and

$$q_I = h_0 w^2 + \frac{g}{2}(h_0 + \eta)^2 - \frac{h_0^3}{3} w_{xt}.$$

Energy conservation

Change of energy E is equal to

- net influx of energy through the boundaries
- plus net work done on the boundary

$$\frac{d}{dt}\mathcal{E} = \left[q_I + \text{work done by pressure force} \right]_{x_2}^{x_1},$$

$$\frac{d}{dt}\int_{x_1}^{x_2}\int_{-h_0}^{\eta} \left\{ \tfrac{1}{2}|\nabla\phi|^2 + g(z+h_0) \right\} dzdx = \left[\int_{-h_0}^{\eta} \left\{ \tfrac{1}{2}|\nabla\phi|^2 + g(z+h_0) \right\}\phi_x \, dz + \int_{-h_0}^{\eta} \phi_x P \, dz \right]_{x_2}^{x_1}$$

Energy conservation

Then the energy balance is

$$\frac{\partial}{\partial \tilde{t}} \tilde{E} + \frac{\partial}{\partial \tilde{x}} \tilde{q}_E = \mathcal{O}(\alpha^2, \alpha\beta, \beta^2).$$

The dimensional forms of the energy density and energy flux are given by

$$E = \tfrac{1}{2} g (h_0 + \eta)^2 + \tfrac{1}{2} h_0 w^2$$

and

$$q_E = g(h_0^2 + 2h_0\eta) w + \tfrac{1}{2} \left(\theta^2 - \tfrac{1}{3} \right) c_0^2 h_0^3 w_{xx}.$$

Application to bores

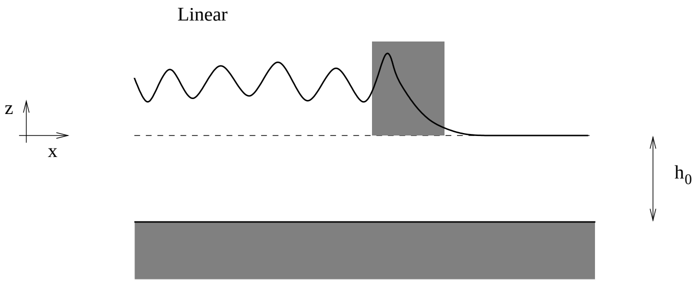
Previous work on undular bore

Favre, Ondes des translation, Dunod, Paris, 1935:
Careful experiments.

Classification of bores into undular, undular with breaking, and turbulent.

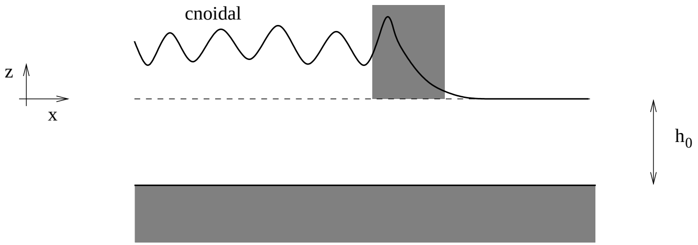
Previous work on undular bore

Lemoine, La Houille Blanche, 1948:
Comparison of energy loss with linear energy flux.



Previous work on undular bore

Benjamin, Lighthill, Proc. Roy. Soc. London A 1954:
"A cnoidal wave-train can be present behind the bore provided that some quanitity of energy intermediate between zero and the classical value is dissipated by friction at the bore itself."



Previous work on undular bore

Benjamin, Lighthill, Proc. Roy. Soc. London A 1954:
"A cnoidal wave-train can be present behind the bore provided that some quantity of energy intermediate between zero and the classical value is dissipated by friction at the bore itself."

KdV equation:

$$u_t + u_x + uu_x + u_{xxx} = 0$$

Normalization:

$$\frac{1}{3}Q^2\left(\frac{d\eta}{dx}\right)^2 + g\eta^3 - 2R\eta^2 + 2S\eta - Q^2 = 0$$

- Q is volume flow rate
- R is energy per unit mass
- S is momentum flow rate

Previous work on undular bore

Sturtevant, Phys. Fluids 1965 :
"the calculations show that there is actually an increase at the front of the bore of both momentum and energy."

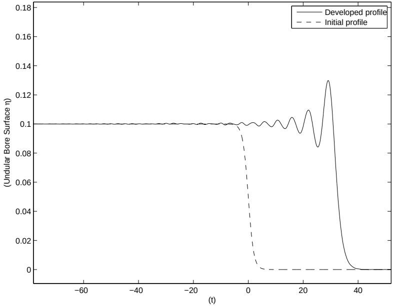
Computations based on Favre's data show that there is a loss of both momentum and energy in the inertial frame of reference.

$\Longrightarrow$ Excess energy is dissipated by bottom boundary layer.

New results

We use the KdV-KdV system

$$\begin{aligned} \eta_t + h_0 w_x + (w\eta)_x + \frac{1}{6} h_0^3\, w_{xxx} &= 0 \\ w_t + g\,\eta_x + w w_x + \frac{1}{6} g h_0^2\, \eta_{xxx} &= 0 \end{aligned}$$



Development of an undular bore with initial amplitude $a_0 = 0.1\,m$

New results

KdV-KdV system

$$\begin{aligned} \eta_t + h_0 w_x + (w\eta)_x + \tfrac{1}{6} h_0^3 w_{xxx} &= 0 \\ w_t + g\,\eta_x + w w_x + \tfrac{1}{6} g h_0^2 \eta_{xxx} &= 0 \end{aligned}$$

Momentum and energy are

$$\mathcal{I} = \int_{x_1}^{x_2} \left\{ (h_0 + \eta) w + \frac{h_0^3}{6} w_{xx} \right\} dx$$

$$\mathcal{E} = \frac{1}{2} \int_{x_1}^{x_2} \left\{ h_0 w^2 + g(h_0^2 + 2 h_0 \eta + \eta^2) \right\} dx$$

New results

BBM-BBM system

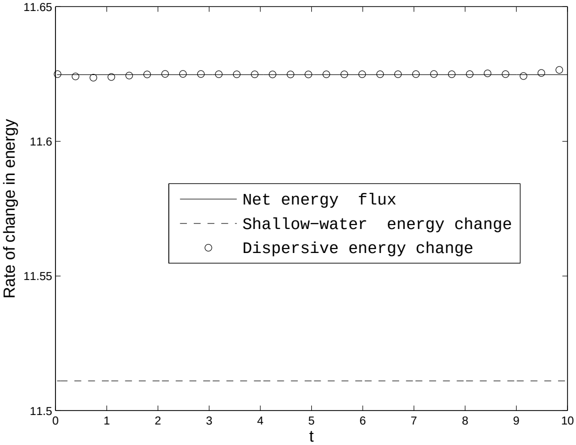
$$\begin{aligned} \eta_t + h_0 w_x + (w\eta)_x - \tfrac{1}{6} h_0^2 \, \eta_{xxt} &= 0 \\ w_t + g\,\eta_x + w w_x - \tfrac{1}{6} h_0^2 \, w_{xxt} &= 0 \end{aligned}$$

Momentum and energy are

$$\mathcal{I} = \int_{x_1}^{x_2} \left\{ (h_0 + \eta) w + \frac{h_0^3}{6} w_{xx} \right\} dx$$

$$\mathcal{E} = \frac{1}{2} \int_{x_1}^{x_2} \left\{ h_0 w^2 + g(h_0^2 + 2 h_0 \eta + \eta^2) \right\} dx$$

New results: Energy conservation



Comparison of the rate of change of energy vs. the net energy flux. The initial bore amplitude was $a_0 = 0.25m$

New results: Energy conservation

a_0	$F_1 - F_2$	$\frac{dE_{sw}}{dt}$	% diff.
0.1	3.64	3.635	0.2
0.2	8.58	8.53	0.6
0.3	15.07	14.88	1.3
0.4	23.35	22.90	1.9
0.5	33.69	32.81	2.6
0.6	46.35	44.86	3.2
0.7	61.63	59.29	3.8
0.8	79.82	76.36	4.3
0.9	101.24	96.35	4.8
1.0	126.20	119.55	5.3

Column 1 shows bore amplitude a_0,
Column 2 shows net energy flux.
Column 3 shows rate of change of energy in shallow-water theory.
Column 4 shows the percentage difference.

New results: Energy conservation

a_0	$F_1 - F_2$	$\frac{dE_{sw}}{dt}$	% diff.	$\frac{dE_{disp}}{dt}$	% diff.
0.1	3.64	3.635	0.2	3.64	0.00
0.2	8.58	8.53	0.6	8.58	0.00
0.3	15.07	14.88	1.3	15.08	0.06
0.4	23.35	22.90	1.9	23.36	0.04
0.5	33.69	32.81	2.6	33.69	0.00
0.6	46.35	44.86	3.2	46.35	0.00
0.7	61.63	59.29	3.8	61.63	0.00
0.8	79.82	76.36	4.3	79.84	0.03
0.9	101.24	96.35	4.8	101.30	0.06
1.0	126.20	119.55	5.3	126.28	0.06

Column 1 shows bore amplitude a_0, note that $h_0 = 1$.
Column 2 shows net energy flux.
Column 3 shows rate of change of energy in shallow-water theory.
Column 4 shows percentage difference.
Column 5 shows rate of change of energy in dispersive theory.

New results: backward velocity case $u_2 = -2ms^{-1}$

Rate of energy change in the the shallow water model compared to the net energy flux.

$\frac{a_0}{h_0}$	$F_1 - F_2$	$\frac{d}{dt}E_{sw}$	diff.
	kgm/s^3	kgm/s^3	%
0.1	0.84	0.83	0.90
0.2	2.19	2.13	2.67
0.3	4.20	4.01	4.64
0.4	7.04	6.58	6.46
0.5	10.87	9.99	8.05
0.6	15.88	14.38	9.41
0.7	22.26	19.92	10.53

New results: backward velocity case $u_2 = -2ms^{-1}$

Rate of energy change in dispersive model compared to the net energy flux.

$\frac{a_0}{h_0}$	$F_1 - F_2$	$\frac{d}{dt}E_{sw}$	diff.	$\frac{d}{dt}E_{disp}$	diff.
	kgm/s^3	kgm/s^3	%	kgm/s^3	%
0.1	0.84	0.83	0.90	0.84	0.00
0.2	2.19	2.13	2.67	2.19	0.01
0.3	4.20	4.01	4.64	4.20	0.02
0.4	7.04	6.58	6.46	7.04	0.03
0.5	10.87	9.99	8.05	10.87	0.04
0.6	15.88	14.38	9.41	15.88	0.05
0.7	22.26	19.92	10.53	22.27	0.05

New results: conservation of momentum

Numerical experiments show that change in momentum is equal to the net influx.

$\frac{a_0}{h_0}$	Net Mom flux	$\frac{d}{dt}M$
	$kgms^{-2}$	$kgms^{-2}$
0.1	1.13	1.13
0.2	2.59	2.59
0.3	4.40	4.40
0.4	6.59	6.59
0.5	9.19	9.19
0.6	12.23	12.23
0.7	15.74	15.74

Applications to KdV equation

KdV equation in dimensional form

$$\eta_t + c_0\eta_x + \frac{3}{2}\frac{c_0}{h_0}\eta\eta_x + \frac{c_0 h_0^2}{6}\eta_{xxx} = 0$$

Mass density:

$$M = h_0 + \eta$$

Mass flux:

$$q_M = c_0\left(\eta + \frac{3}{4h_0}\eta^2 + \frac{h_0^2}{6}\eta_{xx}\right)$$

Momentum density:

$$I = c_0\left(\eta + \frac{3}{4h_0}\eta^2 + \frac{h_0^2}{6}\eta_{xx}\right)$$

Momentum flux:

$$q_I = c_0^2\left(\frac{h_0}{2} + \eta + \frac{3}{2h_0}\eta^2 + \frac{h_0^2}{3}\eta_{xx}\right)$$

KdV equation in dimensional form

Energy density:

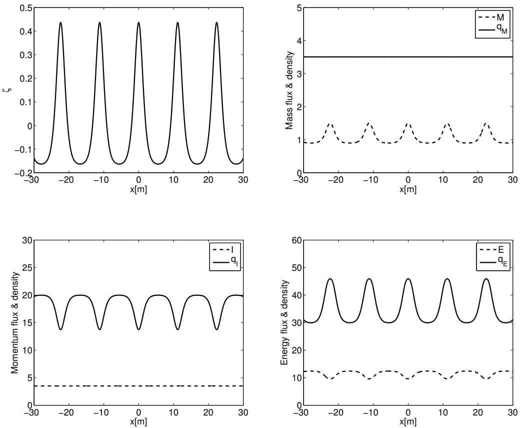
$$E = c_0^2 \left(\frac{h_0}{2} + \eta + \frac{1}{h_0}\eta^2 \right)$$

Energy flux:

$$q_E = c_0^3 \left(\eta + \frac{7}{4h_0}\eta^2 + \frac{h_0^2}{6}\eta_{xx} \right)$$

Total head:

$$H = \frac{g\eta^2}{2h_0} + g(h_0 + \eta)$$

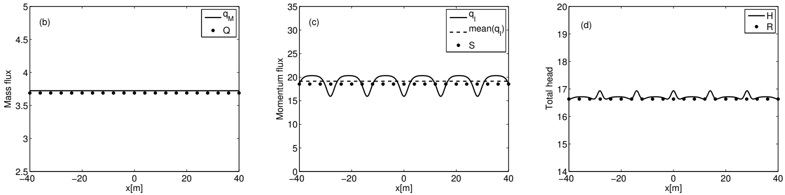


Cnoidal wave $\zeta(x) = \zeta_2 + (\zeta_1 - \zeta_2)\mathrm{cn}^2\left(\sqrt{\frac{3(\zeta_1-\zeta_3)}{4h_0^3}}x; m\right)$

$h_0 = 1.0631$, $\zeta_1 = 0.4369$, $\zeta_2 = -0.1631$, and $\zeta_3 = -0.1731$

Steady KdV equation in traveling reference frame:

$$\frac{1}{3}Q^2\left(\frac{d\eta}{dx}\right)^2 + g\eta^3 - 2R\eta^2 + 2S\eta - Q^2 = 0$$

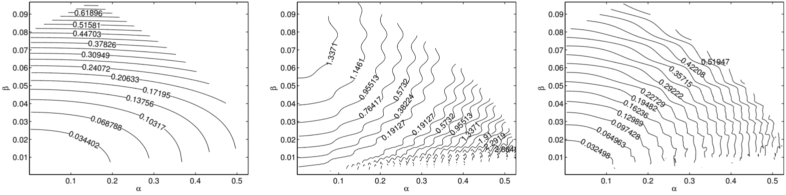


(b) mass flux q_M vs. Q

(c) momentum flux q_M vs. S

(d) total head H vs. R

$h_0 = 1.1$, $\zeta_1 = 0.3$, $\zeta_2 = -0.1$ and $\zeta_3 = -0.15$



(a) $|\overline{q_M} - Q|$ (b) $|\overline{q_I} - S|$ (c) $|\overline{H} - R|$

Errors in the approximation of Q, R, S in terms of α and β

Integrable models

Integrable models

Kaup system:

$$\eta_t + h_0 w_x + (\eta w)_x + \frac{h_0^3}{3} w_{xxx} = 0,$$
$$w_t + g\eta_x + w w_x = 0,$$

Hamiltonian:

$$\mathcal{H}_{Kaup} = \int_{-\infty}^{\infty} \left\{ \tfrac{g}{2}\eta^2 + \tfrac{1}{2}(h_0 + \eta) w^2 - \tfrac{h_0^3}{6} w_x^2 \right\} dx$$

KdV equation:

$$\eta_t + c_0 \eta_x + \frac{3}{2}\frac{c_0}{h_0}\eta\eta_x + \frac{c_0 h_0^2}{6}\eta_{xxx} = 0$$

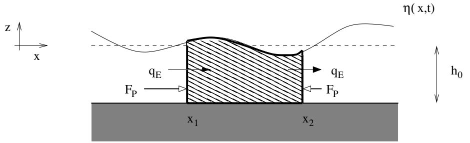
Hamiltonian:

$$\mathcal{H}_{KdV} = \int_{-\infty}^{\infty} \left\{ \tfrac{1}{3} h_0^3 \eta_x^2 - \eta^3 \right\} dx$$

Energy conservation

Different normalization of potential energy:

...a particle located at the level of the undisturbed free surface has zero potential energy, and the total potential energy is zero when no wave motion is present ...



$$\frac{d}{dt}\int_{x_1}^{x_2}\int_{-h_0}^{\eta}\left\{\tfrac{1}{2}|\nabla\phi|^2 + gz\right\} dzdx =$$

$$\left[\int_{-h_0}^{\eta}\left\{\tfrac{1}{2}|\nabla\phi|^2 + gz\right\}\phi_x\, dz + \int_{-h_0}^{\eta}\phi_x P\, dz\right]_{x_2}^{x_1}$$

Energy conservation for Kaup system

Then the energy balance is

$$\frac{\partial}{\partial \tilde{t}}\tilde{E} + \frac{\partial}{\partial \tilde{x}}\tilde{q}_E = \mathcal{O}(\alpha^2, \alpha\beta, \beta^2).$$

The dimensional forms of the energy density and energy flux are given by

$$E = \rho\left\{\frac{g}{2}\eta^2 + \frac{1}{2}(h_0+\eta)w^2 + \frac{h_0^3}{3}ww_{xx} + \frac{h_0^3}{6}w_x^2\right\}$$

and

$$q_E = \rho\left\{\frac{h_0}{2}w^3 + g\eta w(h_0+\eta) + \frac{gh_0^3}{3}\eta w_{xx} - \frac{h_0^3}{3}ww_{xt}\right\}$$

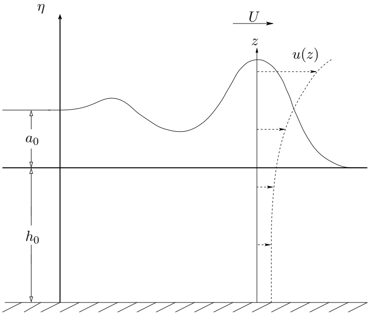
We have

$$\mathcal{H}_{Kaup} = \int_{-\infty}^{\infty} E(x,t)\,dx$$

Breaking criterion for undular bores in Boussinesq theory

Velocity profile

$$u(x,z,t) = \phi_x(x,z,t) \approx w(x,t) + \frac{1}{2}\left((h_0\theta)^2 - z^2\right) w_{xx}(x,t)\,.$$

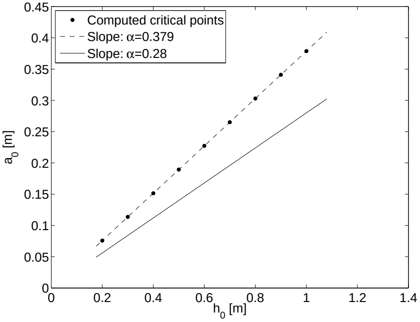


the bottom, yields

Breaking criterion

A wave solution $(w(x,t), \eta(x,t))$ *starts to break if*

$$w(x,t) + \frac{1}{2}\left\{h_0^2\theta^2 - \left(h_0 + \eta(x,t)\right)^2\right\} w_{xx}(x,t) > U.$$



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