The Cauchy problem for the CH and DP equations

Alex Himonas

University of Notre Dame, USA

June 26, 2012

Nonlinear waves and interface problems

Centre for Mathematical Sciences, Lund University

June 26 – 28, 2012

Abstract

We shall consider the initial value problem for the Camassa-Holm (CH) and Degasperis-Procesi (DP) equations and discuss their well-posedness properties in Sobolev spaces H^s. For $s > 3/2$ these equations are well-posed and their data-to-solution map is continuous but not uniformly continuous. When $s < 3/2$, both CH and DP are ill-posed in Sobolev spaces H^s, and this will be the main focus of our presentation. The talk is based on work with Carlos Kenig, Gerard Misiołek, Curtis Holliman and Katelyn Grayshan.

References

J. L. Bona and N. Tzvetkov, *Sharp well-posedness results for the BBM equation*. Discrete Contin. Dyn. Syst. **23** (2009), no. 4, 1241–1252.

A. Bressan and A. Constantin, *Global conservative solutions of the Camassa-Holm equation*, Arch. Rat. Mech. Anal. **183** (2007), 215–239.

A. Bressan and A. Constantin, *Global dissipative solutions of the Camassa-Holm equation*, Anal. Appl. **5** (2007), no. 1, 1–27.

R. Camassa and D. Holm, *An integrable shallow water equation with peaked solitons*. Phys. Rev. Lett. **71** (1993), 1661–1664.

M. Christ, J. Colliander and T. Tao, *Asymptotics, frequency modulation, and low regularity ill-posedness for canonical defocusing equations*. Amer. J. Math. **125** (2003), no. 6, 1235–1293.

O. Christov and S. Hakkaev, *On the Cauchy problem for the periodic b-family of equations and of the non-uniform continuity of the Degasperis-Procesi equation*. J. Math. Anal. Appl. **360** (2009), no.1, 47-56.

References (cont.)

A. Constantin and J. Escher, *Well-posedness, global existence and blowup phenomena for a periodic quasi-linear hyperbolic equation*, Comm. Pure Appl. Math. **51** (1998), 475–504.

A. Constantin and D. Lannes, *The hydrodynamical relevance of the Camassa-Holm and Degasperis-Procesi equations*, Arch. Ration. Mech. Anal. **192** (2009), no. 1, 165–186.

A. Constantin, *On the Cauchy problem for the periodic Camassa-Holm equation*, J. Differential Equations **141** (1997).

A. Constantin and J. Escher, *Well-posedness, global existence and blowup phenomena for a periodic quasi-linear hyperbolic equation*, Comm. Pure Appl. Math. **51** (1998).

A. Constantin and J. Escher, *Global weak solutions for a shallow water equation*, Indiana Univ. Math. J. **47** (1998).

A. Constantin and H. McKean, *A shallow water equation on the circle*, Comm. Pure Appl. Math. **52** (1999).

R. Danchin, *A few remarks on the Camassa-Holm equation*, Differential Integral Equations **14** (2001), no. 8, 953–988.

References (cont.)

A. Degasperis, D. D. Holm and A. N. W. Hone, *A new integral equation with peakon solutions*. Theoret. Math. Phys. **133** (2002), 1463-1474.

A. Degasperis, D. Holm and A. Hone, *A new integrable equation with peakon solutions*. Theoret. and Math. Phys. 133 (2002), no. 2, 14631474.

A. Degasperis and M. Procesi, *Asymptotic integrability symmetry and perturbation theory*. (Rome, 1998), World Sci. Publ., (1999), 23–37.

D. Ebin and J. Marsden, *Groups of diffeomorphisms and the motion of an incompressible fluid*. Ann. Math. **92** (1970), pp. 102–163.

J. Escher, Y. Liu and Z. Yin, *Shock waves and blow-up phenomena for the periodic Degasperis-Procesi equation*. Indiana Univ. Math. J. **56** (2007), no. 1, 87–117.

J. Escher, Y. Liu and Z. Yin, *Global weak solutions and blow-up structure for the Degasperis-Procesi equation*. J. Funct. Anal. **241** (2006), no. 2, 257–485.

A. Fokas, *On a class of physically important integrable equations.* The nonlinear Schrdinger equation (Chernogolovka, 1994). Phys. D **87** (1995), no. 1-4, 145150.

A. Fokas and B. Fuchssteiner, *Symplectic structures, their Bäklund transformations and hereditary symmetries*. Phys. D **4** (1981/82), no. 1, 47–66.

A. Himonas and C. Kenig, *Non-uniform dependence on initial data for the CH equation on the line*. Differential and Integral Equations, **22**, No. 3-4, (2009), 201– 224.

A. Himonas, C. Kenig and G. Misiołek, *Non-uniform dependence for the periodic CH equation*. Comm. Partial Differential Equations, to appear.

A. Himonas and G. Misiołek, *High-frequency smooth solutions and well-posedness of the Camassa-Holm equation*. Int. Math. Res. Not. **51** (2005), 3135-3151.

A. Himonas and G. Misiołek, *Non-uniform dependence on initial data of solutions to the Euler equations of hydrodynamics*. Comm. Math. Phys., (2010).

A. Himonas and C. Holliman, *On well-posedness of the Degasperis-Procesi equation*. Preprint (2010).

D. Holm and A. Hone, *A class of equations with peakon and pulson solutions.* (With an appendix by H. Braden and J. Byatt-Smith) J. Nonlinear Math. Phys. **12** (2005), suppl. 1, 380394

A. Hone, H. Lundmark and J. Szmigielski, *Explicit multipeakon solutions of Novikov's cubically nonlinear integrable Cammasa-Holm type equation.* Dynamics of PDE **6** (2009), no. 3, 253–289.

A. Hone and J. P. Wang, *Integrable peakon equations with cubic nonlinearity.* Preprint (2009).

D. Karapetyan *Non-uniform dependence and well-posedness for the hyperelastic rod equation*, J. Differential Equations **249** (2010) 796–826.

T. Kato and G. Ponce, *Commutator estimates and the Euler and Navier-Stokes equations*. Comm. Pure Appl. Math. **41** (1988), no. 7, pp. 891-907.

C. Kenig, G. Ponce and L. Vega, *On the ill-posedness of some canonical dispersive equations*. Duke Math. J. **106**, no. 3, (2001), 617–633.

References (cont.)

H. Koch and N. Tzvetkov, *Nonlinear wave interactions for the Benjamin-Ono equation*. Int. Math. Res. Not. **30** (2005), 1833-1847.

J. Lenells, *Traveling wave solutions of the Degasperis-Procesi equation*. J. Math. Anal. Appl. **306** (2005), no. 1, 72-82.

Y. Li and P. Olver, *Well-posedness and blow-up solutions for an integrable nonlinearly dispersive model wave equation*. J. Differential Equations **162** (2000), pp. 27-63.

A. Mikhailov and V. Novikov, *Perturbative symmetry approach*. J. Phys. A **35** (2002), no. 22, 47754790.

Z. Qiao, *New integrable hierarchy, its parametric solutions, cuspons, one-peak solitons, and M/W-shape peak solitons.* J. Math. Phys. **48** (2007), no. 8, 08270, 20 pp.

F. Tiglay, *The Periodic Cauchy Problem for Novikov's Equation*. arxiv:1009.1820 (2010).

M. Taylor, *Commutator estimates*. Proc. Amer. Math. Soc. 131 (2003), no. 5, 1501–1507.

Solitary wave in Hawaii (Photo by Odom)

Braking wave (Wikipedia)

Water wave equations

Can the **motions of such a wave** be modeled by a solution to an evolution equation like the Korteweg-de Vries (KdV) equation

$$\partial_t u + u\partial_x u + \partial_x^3 u = 0,$$

or the Camassa-Holm (CH) equation

$$u_t - u_{xxt} + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0\,?$$

CH type equations [Novikov, 2009]

• In [J. Phys. A, 2009] Novikov investigated the question of **integrability** for equations of the form

$$(1 - \partial_x^2)u_t = F(u, u_x, u_{xx}, u_{xxx}, \cdots), \tag{1}$$

where F is a polynomial of u and its x-derivatives.

• **Definition of integrability:** Existence of an infinite hierarchy of (quasi-) local higher symmetries.

CH type equations [Novikov, 2009] (cont.)

He produced about 20 CH equations with quadratic nonlinearities that include the following two:

1• Camassa-Holm (CH): $(1-\partial_x^2)u_t = -3uu_x + 2u_x u_{xx} + uu_{xxx}$

2• Degasperis-Procesi (DP): $(1-\partial_x^2)u_t = -4uu_x + 3u_x u_{xx} + uu_{xxx}$

...

and about 10 CH equations with cubic nonlinearities including the following one:

3• Novikov equation (NE): $(1-\partial_x^2)u_t = -4u^2u_x + 3uu_x u_{xx} + u^2 u_{xxx}$

...

Fokas-Fuchssteiner derivation

If for every n the operator $\theta_1 + n\theta_2$ is Hamiltonian, then

$$q_t = -(\theta_2\theta_1^{-1})q_x, \tag{2}$$

is an integrable equation. Letting $\theta_1 = \partial \doteq \partial_x$, and

$$\theta_2 = \partial + \gamma\partial^3 + \frac{\alpha}{3}(q\partial + \partial q), \quad \alpha, \gamma \text{ constants},$$

gives the celebrated KdV equation

$$q_t + u_x + \gamma q_{xxx} + \alpha q q_x = 0. \tag{3}$$

Similarly, letting $\theta_1 = \partial + \nu\partial^3$ and θ_2 as above with $q = u + \nu u_{xx}$ gives the CH equation

$$u_t + u_x + \nu u_{xxt} + \gamma u_{xxx} + \alpha u u_x + \frac{\alpha\nu}{3}(u u_{xxx} + 2u_x u_{xx}) = 0. \tag{4}$$

• This equation was derived physically as a **water wave** equation by Camassa and Holm (1993) who also studied its "peakon" solutions.

Degasperis-Procesi derivation

This equation was introduced in the 1999 paper of Degasperis and Procesi, *Asymptotic integrability symmetry and perturbation theory*, as one of three equations to satisfy asymptotic integrability conditions for the following familty of third order dispersive PDE conservation laws

$$u_t + c_0 u_x + \gamma u_{xxx} - \alpha^2 u_{txx} = (c_1 u^2 + c_2 u_x^2 + c_3 u u_{xx})_x, \qquad (5)$$

where $\alpha, c_0, c_1, c_2, c_3 \in \mathbb{R}$ are fixed constants. The other two equations arriving out of this machinery are the Camassa-Holm and KdV equations.

Common phenomenon of CH, DP and NE

- They all have **peakon** solutions:

$$u(x,t) = ce^{-|x-ct|}$$

- The discovery of the CH equation by Camassa-Holm [1993] (Fokas-Fuchssteiner 1981) was partly driven by the desire to find a water wave equation which has traveling wave solutions that break. The Korteweg-de Vries equation (KdV),

$$\partial_t u + 6u\partial_x u + \partial_x^3 u = 0, \tag{6}$$

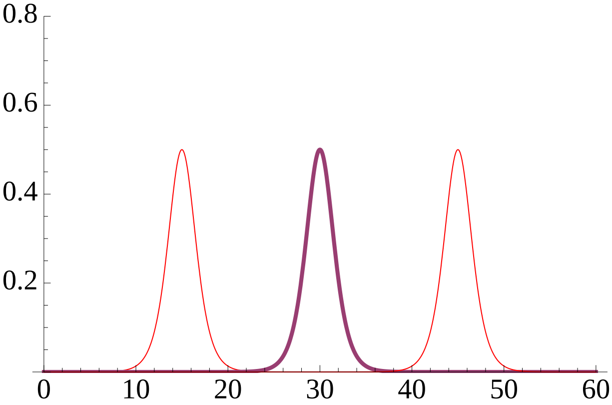
which was derived in 1895 as a model of long water waves propagating in a channel has only **smooth solitons**.

- Also, CH, DP and NE have **multipeakon** solutions:

$$u(x,t) = \sum_{j=1}^{n} p_j(t) e^{-|x-q_j(t)|}$$

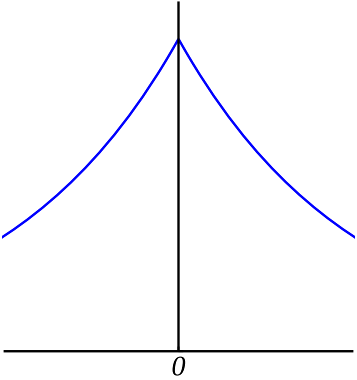
KdV Soliton: $u(x,t) = f(x - ct)$

$$f(x) = \frac{c}{2}\text{sech}^2\left(\frac{\sqrt{c}}{2}\,x\right)$$



Peakon: $u(x,t) = f(x-ct)$

CH & DP: $f(x) = ce^{-|x|}$ NE: $f(x) = \sqrt{c}e^{-|x|}$



Definition of Well-Posedness in H^s

(i) For any initial data $u(0) \in H^s$ there exists $T = T_{u(0)} > 0$ and a solution $u \in C([0, T]; H^s)$ to the DP (CH or NE) Cauchy problem.

(ii) This solution u is unique in the space $u \in C([0, T]; H^s)$.

(iii) The data-to-solution map $u(0) \mapsto u(t)$ is continuous. More precisely, if $u_n(0)$ is a sequence of initial data converging to $u_\infty(0)$ in H^s and if $u_n(t) \in C([0, T_n]; H^s)$ is the solution to the Cauchy problem with initial data $u_n(0)$, then there is $T \in (0, T_\infty)$ such that the solutions $u_n(t)$ can be extended to the interval $[0, T]$ for all sufficiently large n and

$$\lim_{n\to\infty} \sup_{0\leq t\leq T} \|u_n(t) - u_\infty(t)\|_{H^s} = 0. \tag{7}$$

Sharp Well-Posedness for CH and DP

Theorem (Well-Posedness)

1• If $s > 3/2$ and $u_0 \in H^s$ then there exists $T > 0$ and a unique solution $u \in C([0, T]; H^s)$ of the initial value problem for CH or DP which depends continuously on the initial data u_0. Furthermore, we have the estimate

$$\|u(t)\|_{H^s} \leq 2\|u_0\|_{H^s}, \quad \text{for} \quad 0 \leq t \leq T \leq 1/(2c_s\|u_0\|_{H^s}) \tag{8}$$

where $c_s > 0$ is a constant depending on s.
2• Also, the data-to-solution map is not uniformly continuous from any bounded subset in H^s into $C([0, T]; H^s)$.

- CH equation (H., Kenig, Misiolek,)
- DP equation (H., Holliman)

The Proof of well-posedness

- The proof of **(1)** (well-posedness) begins by solving the mollified i.v.p.

$$\partial_t u + J_\varepsilon\Big[(J_\varepsilon u)\partial_x(J_\varepsilon u)\Big] + F(u) = 0, \quad u(x,0) = u_0(x), \tag{9}$$

with

$$F(u) = -\partial_x(1-\partial_x^2)^{-1}\Big[\frac{b}{2}u^2 + \frac{3-b}{2}(\partial_x u)^2\Big]$$

where for CH $b = 2$ and DP $b = 3$.

...

The Proof of nonuniform continuity

We prove that there exist two sequences of CH or DP solutions $u_n(t)$ and $v_n(t)$ in $C([0, T]; H^s(\mathbb{R}))$ such that:

1• $$\sup_n \|u_n(t)\|_{H^s} + \sup_n \|v_n(t)\|_{H^s} \lesssim 1,$$

2• $$\lim_{n\to\infty} \|u_n(0) - v_n(0)\|_{H^s} = 0,$$

3• $$\liminf_n \|u_n(t) - v_n(t)\|_{H^s} \gtrsim \sin t, \quad 0 \le t < T \le 1,$$

For this we use approximate solutions.

Approximate solutions for CH and DP

- The approximate solutions

$$u^{\omega,n}(x,t) = \omega n^{-1} + n^{-s}\cos(nx - \omega t), \quad \text{for} \quad \omega = -1, 1, \qquad (10)$$

where $n \in \mathbb{Z}^+$, satisfy conditions (1)-(3) for non-uniform dependence of the periodic CH and DP equations **but they are not solutions.**

- However, the **error**

$$E = CH(u^{\omega,n}) \quad \text{or} \quad E = DP(u^{\omega,n})$$

is **small**.

- Then solving the Cauchy problem with initial data

$$u^{\omega,n}(x,0)$$

gives actual solutions, which satisfy the three condition of the non-uniform continuity.

Remark. For the Benjamin-Ono equation this **method** was used by H. Koch and N. Tzvetkov in IMRN (2005).

Well-Posedness for the Novikov equation

Theorem (H.–Holliman)

1• If $s > 3/2$ and $u_0 \in H^s$ then there exists $T > 0$ and a unique solution $u \in C\big([0, T]; H^s\big)$ of the initial value problem for NE, which depends continuously on the initial data u_0. Furthermore, we have the estimate

$$\|u(t)\|_{H^s} \leq 2\|u_0\|_{H^s}, \quad \text{for} \quad 0 \leq t \leq T \leq \frac{1}{4c_s\|u_0\|_{H^s}^2}, \tag{11}$$

where $c_s > 0$ is a constant depending on s.

2• Also, the data-to-solution map is not uniformly continuous from any bounded subset in H^s into $C([0, T]; H^s)$.

The Proof for NE

- The proof of **(1)** (well-posedness) begins by solving the mollified i.v.p.

$$\partial_t u + J_\varepsilon\Big[(J_\varepsilon u)^2 \partial_x (J_\varepsilon u)\Big] + F(u) = 0, \quad u(x,0) = u_0(x), \qquad (12)$$

with

$$F(u) = -(1-\partial_x^2)^{-1}\partial_x\Big[u^3 - \frac{3}{2}\Big(u(\partial_x u)^2\Big)\Big] - (1-\partial_x^2)^{-1}\Big[\frac{1}{2}(\partial_x u)^3\Big].$$

- The NE approximate solutions used for the proof of **(2)** on the circle are:

$$u^{\omega,n} = \omega n^{-1/2} + n^{-s}\cos(nx - \omega t), \quad \text{for} \quad \omega = 0, 1. \qquad (13)$$

Note. Unlike in the case of CH and DP one of them ($\omega = 0$) has **no low frequency**.

Approximate solutions motivation (from Euler eqns)

- For any $\omega \in \mathbb{R}$ and $n \in \mathbb{Z}^+$ the divergence free vector field

$$u^{\omega,n}(t,x) = \big(\omega n^{-1} + n^{-s}\cos(nx_2 - \omega t),\, \omega n^{-1} + n^{-s}\cos(nx_1 - \omega t)\big)$$

is a solution! to the Euler equations on $\mathbb{T}^2$.

- The corresponding to $\omega \pm 1$ sequences

$$u^{+1,n}(t,x) \qquad \text{and} \qquad u^{-1,n}(t,x)$$

Satisfy conditions (1)-(3) for non-uniform dependence of the periodic Euler equations in 2-D.

Remark. Non-uniform dependence for the Euler equations in n-D was proved in [H.–Misiołek, CMP 2010].

Non-uniform dependence for the Euler equations

Theorem (H.–Misiołek, CMP 2010)

Let $s \in \mathbb{R}$. The solution map $u_0 \mapsto u$ of the Cauchy problem for the Euler equations

$$\partial_t u + \nabla_u u + \nabla p = 0 \tag{14}$$

$$\operatorname{div} u = 0$$

$$u(0, x) = u_0(x), \quad x \in \mathbb{T}^n, \ t \in \mathbb{R} \tag{15}$$

is **not uniformly continuous** *from any bounded subset in $H^s(\mathbb{T}^n, \mathbb{R}^n)$ into $C\big([0, T], H^s(\mathbb{T}^n, \mathbb{R}^n)\big)$.*
(Here $n = 2$ or 3 and T is the lifespan.)
The same is true $\mathbb{R}^n$ if $s > 0$

I$\ell\ell$-posedness for CH and DP

Theorem (H., Holliman, Grayshan, Byers)

For $s < 3/2$ the Cauchy problem for CH and DP is not well-posed in H^s in the sense of Hadamard.

The Proof is based on:

- Conserved quantities (H^1 for CH and twisted L^2 for DP).
- The interaction of peakon-antipeakon traveling wave solutions.

Next, we describe the **proof for the case of DP on the line**.

Nonlocal form of DP

The Cauchy problem for the Degasperis-Procesi (DP) equation, written in its nonlocal form, is given by

$$u_t + uu_x + \frac{3}{2}\partial_x D^{-2}\left[u^2\right] = 0, \quad D^{-2} \doteq (1-\partial_x^2)^{-1}, \tag{16}$$

$$u(x,0) = u_0(x). \tag{17}$$

A Conserved Quantity for DP

The first tool used in our proof of Theorem 4 is the conservation of the following quantity:

$$\begin{aligned}
\|f\|_{\tilde{L}^2}^2 &\doteq \int_{\mathbb{R}} (1-\partial_x^2)f \cdot (4-\partial_x^2)^{-1} f(x)dx \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} \frac{1+\xi^2}{4+\xi^2}|\widehat{f}(\xi)|^2 d\xi \\
&\simeq \|f\|_{L^2}^2 .
\end{aligned} \tag{18}$$

Peakon-antipeakon solutions

The peakon-antipeakon traveling wave

$$u(x,t) = p(t)e^{-|x+q(t)|} - p(t)e^{-|x-q(t)|} \quad (19)$$

is a weak solution to the DP equation if the amplitude $p = p(t)$ and the "speed" $q = q(t)$ are solutions to the following system of ordinary differential equations

$$q' = p(e^{-2q} - 1) \text{ and } p' = 2p^2 e^{-2q}. \quad (20)$$

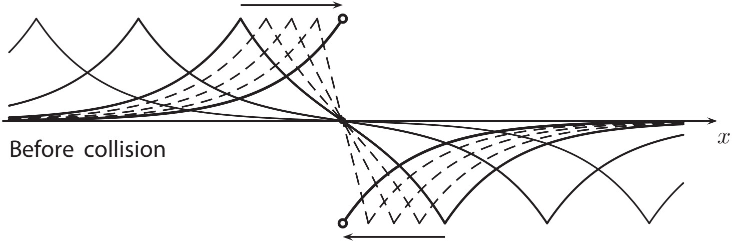
Furthermore, if for given $\varepsilon > 0$ we choose p_0 and q_0 so that

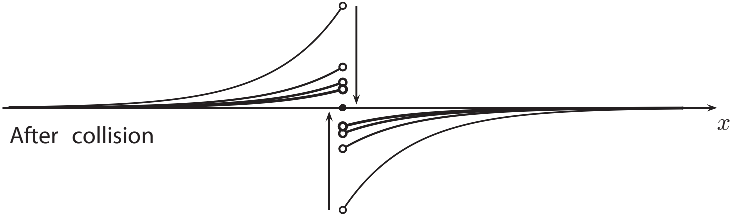
$$p_0 \geq \frac{1}{\varepsilon} \quad \text{and} \quad q_0 < \frac{\ln 2}{8}, \quad (21)$$

there exist $\boxed{0 < T < \varepsilon}$ such that the system of ODEs has a unique smooth solution $(q(t), p(t))$ in the interval $[0, T)$ with $p(t)$ increasing, $q(t)$ decreasing, and

$$\lim_{t \to T^-} p(t) = \infty \quad \text{and} \quad \lim_{t \to T^-} q(t) = 0. \quad (22)$$

DP Peakon-antipeakon collision





The Fundamental Estimate

Lemma

Let $u(t)$ be the the peakon-antipeakon solution, then for all $t \in [0, T)$ we have the estimates

$$\|u(t)\|_{H^s} \approx \begin{cases} p(t)q(t)^{3/2-s}, & 1/2 < s < 3/2, \\ p(t)q(t)\sqrt{\ln(1/q(t))}, & s = 1/2, \\ p(t)q(t), & s < 1/2. \end{cases} \tag{23}$$

Norm Inflation for DP

Using the last estimate and the conservation of the $\tilde{L}^2$-norm gives the following result.

Theorem

(Norm Inflation) Let $s \in [\frac{1}{2}, \frac{3}{2})$. *Then for any* $\varepsilon > 0$ *there exists* $T > 0$ *such that the DP Cauchy problem has a solution* $u \in C([0, T); H^s)$ *satisfying the following three properties:*

(1) *Lifespan* $T < \varepsilon$,

(2) $\|u_0\|_{H^s} < \varepsilon$,

(3) $\lim_{t \to T^-} \|u(t)\|_{H^s} = \infty$, *(norm inflation).*

Note. For CH norm inflation occurs when $s \in (1, \frac{3}{2})$

Proof of DP Ill-Posedness Theorem: $1/2 \leq s < 3/2$

- Let $u_n(t)$ be the peakon-antipeakon DP solution corresponding to the choice of $\varepsilon = 1/n$ and let $u_\infty(t) = 0$. Then, by property (2) in Norm Inflation result we have $\|u_n(0)\|_{H^s} < 1/n$. Therefore, $u_n(0)$ converges to $u_\infty(0) = 0$ in H^s.
- Furthermore, by property (1) the lifespan T_n of each solution $u_n(t)$ satisfies the inequality $T_n < 1/n$, whereas the lifespan T_∞ of $u_\infty(t)$ is equal to ∞.
- Since, by property (3) in Norm Inflation result there is no $T > 0$ such that the solutions $u_n(t)$ can be extended to the interval $[0, T]$ for all sufficiently large n we see that continuity condition (iii) of well-posedness fails. $\square$

Proof of Ill-Posedness Theorem: $s < 1/2$

Now, the H^s limit of $u(t)$ as $t \nearrow T$ exists and is given by

$$u(x, T) = -\sigma(x) p_0 (1 - e^{-2q_0}) e^{-|x|}, \tag{24}$$

where $\sigma(x)$ is the sign function. Also, the shock peakon

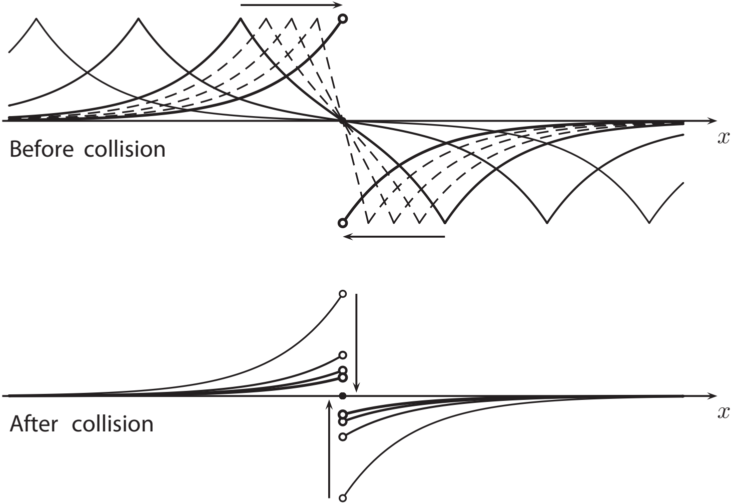
$$v(x, t) = \frac{\sigma(x)}{(T - t) - [p_0(1 - e^{-2q_0})]^{-1}} e^{-|x|} \tag{25}$$

is a solution to the DP equation for $t \neq T - [p_0(1 - e^{-2q_0})]^{-1}$, and $v(t) \in H^s$ if $s < 1/2$. Furthermore, we have

$$T = q_0 [p_0(1 - e^{-2q_0})]^{-1}, \tag{26}$$

which implies that $T - [p_0(1 - e^{-2q_0})]^{-1} = (q_0 - 1)[p_0(1 - e^{-2q_0})]^{-1} < 0$. Therefore, the shock peakon solution $v(t)$ is well-defined for all $t \geq 0$ and an element of $C([0, T]; H^s)$. This together with the fact that $u(T) = v(T)$ shows that uniqueness of solutions for DP fails.

DP Peakon-antipeakon collision [Lundmark, 2006]



Norm Inflation: Proof for $1/2 < s < 3/2$

We use the fact that u conserves the $\tilde{L}^2$-norm, that is to say for some C we have $C = \|u(t)\|_{\tilde{L}^2}$. Therefore, using the fundamental estimate and the conserved quantity $\tilde{L}^2$

$$\frac{C}{\|u(t)\|_{H^s}} = \frac{\|u(t)\|_{\tilde{L}^2}}{\|u(t)\|_{H^s}} \leq \frac{\|u(t)\|_{H^0}}{\|u(t)\|_{H^s}} \leq \frac{c_0 pq}{c_s^{-1} pq^{3/2-s}} = c_0 c_s q^{s-1/2}. \quad (27)$$

Since $s > 1/2$ and $q \to 0$ we obtain

$$0 \leq \lim_{t \to T} \frac{C}{\|u(t)\|_{H^s}} \leq c_0 c_s \lim_{t \to T} q^{s-1/2} = 0, \quad (28)$$

Norm Inflation: Proof for $s = 1/2$

The case for $s = 1/2$ follows in the same way

$$\frac{C}{\|u(t)\|_{H^{1/2}}} \leq \frac{\|u(t)\|_{H^0}}{\|u(t)\|_{H^{1/2}}} \leq \frac{c_0 pq}{c_{1/2}^{-1} pq \sqrt{\ln(1/q)}} = \frac{c_0 c_{1/2}}{\sqrt{\ln(1/q)}}, \tag{29}$$

which yields

$$0 \leq \lim_{t \to T} \frac{C}{\|u(t)\|_{H^{1/2}}} \leq \lim_{t \to T} \frac{c_0 c_{1/2}}{\sqrt{\ln(1/q)}} = 0, \tag{30}$$

or $\lim_{t \to T} \|u(t)\|_{H^{1/2}} = \infty$.

Remark. Norm inflation for CH holds when $1 < s < 3/2$ (Byers).

Thank you!

References

J. L. Bona and N. Tzvetkov, *Sharp well-posedness results for the BBM equation*. Discrete Contin. Dyn. Syst. **23** (2009), no. 4, 1241–1252.

A. Bressan and A. Constantin, *Global conservative solutions of the Camassa-Holm equation*, Arch. Rat. Mech. Anal. **183** (2007), 215–239.

A. Bressan and A. Constantin, *Global dissipative solutions of the Camassa-Holm equation*, Anal. Appl. **5** (2007), no. 1, 1–27.

R. Camassa and D. Holm, *An integrable shallow water equation with peaked solitons*. Phys. Rev. Lett. **71** (1993), 1661–1664.

M. Christ, J. Colliander and T. Tao, *Asymptotics, frequency modulation, and low regularity ill-posedness for canonical defocusing equations*. Amer. J. Math. **125** (2003), no. 6, 1235–1293.

O. Christov and S. Hakkaev, *On the Cauchy problem for the periodic b-family of equations and of the non-uniform continuity of the Degasperis-Procesi equation*. J. Math. Anal. Appl. **360** (2009), no.1, 47-56.

References (cont.)

A. Constantin and J. Escher, *Well-posedness, global existence and blowup phenomena for a periodic quasi-linear hyperbolic equation*, Comm. Pure Appl. Math. **51** (1998), 475–504.

A. Constantin and D. Lannes, *The hydrodynamical relevance of the Camassa-Holm and Degasperis-Procesi equations*, Arch. Ration. Mech. Anal. **192** (2009), no. 1, 165–186.

A. Constantin, *On the Cauchy problem for the periodic Camassa-Holm equation*, J. Differential Equations **141** (1997).

A. Constantin and J. Escher, *Well-posedness, global existence and blowup phenomena for a periodic quasi-linear hyperbolic equation*, Comm. Pure Appl. Math. **51** (1998).

A. Constantin and J. Escher, *Global weak solutions for a shallow water equation*, Indiana Univ. Math. J. **47** (1998).

A. Constantin and H. McKean, *A shallow water equation on the circle*, Comm. Pure Appl. Math. **52** (1999).

R. Danchin, *A few remarks on the Camassa-Holm equation*, Differential Integral Equations **14** (2001), no. 8, 953–988.

References (cont.)

A. Degasperis, D. D. Holm and A. N. W. Hone, *A new integral equation with peakon solutions*. Theoret. Math. Phys. **133** (2002), 1463-1474.

A. Degasperis, D. Holm and A. Hone, *A new integrable equation with peakon solutions*. Theoret. and Math. Phys. 133 (2002), no. 2, 14631474.

A. Degasperis and M. Procesi, *Asymptotic integrability symmetry and perturbation theory*. (Rome, 1998), World Sci. Publ., (1999), 23–37.

D. Ebin and J. Marsden, *Groups of diffeomorphisms and the motion of an incompressible fluid*. Ann. Math. **92** (1970), pp. 102–163.

J. Escher, Y. Liu and Z. Yin, *Shock waves and blow-up phenomena for the periodic Degasperis-Procesi equation*. Indiana Univ. Math. J. **56** (2007), no. 1, 87–117.

J. Escher, Y. Liu and Z. Yin, *Global weak solutions and blow-up structure for the Degasperis-Procesi equation*. J. Funct. Anal. **241** (2006), no. 2, 257–485.

A. Fokas, *On a class of physically important integrable equations.* The nonlinear Schrdinger equation (Chernogolovka, 1994). Phys. D **87** (1995), no. 1-4, 145150.

References (cont.)

A. Fokas and B. Fuchssteiner, *Symplectic structures, their Bäklund transformations and hereditary symmetries*. Phys. D **4** (1981/82), no. 1, 47–66.

A. Himonas and C. Kenig, *Non-uniform dependence on initial data for the CH equation on the line*. Differential and Integral Equations, **22**, No. 3-4, (2009), 201– 224.

A. Himonas, C. Kenig and G. Misiołek, *Non-uniform dependence for the periodic CH equation*. Comm. Partial Differential Equations, to appear.

A. Himonas and G. Misiołek, *High-frequency smooth solutions and well-posedness of the Camassa-Holm equation*. Int. Math. Res. Not. **51** (2005), 3135-3151.

A. Himonas and G. Misiołek, *Non-uniform dependence on initial data of solutions to the Euler equations of hydrodynamics*. Comm. Math. Phys., (2010).

A. Himonas and C. Holliman, *On well-posedness of the Degasperis-Procesi equation*. Preprint (2010).

References (cont.)

D. Holm and A. Hone, *A class of equations with peakon and pulson solutions.* (With an appendix by H. Braden and J. Byatt-Smith) J. Nonlinear Math. Phys. **12** (2005), suppl. 1, 380394

A. Hone, H. Lundmark and J. Szmigielski, *Explicit multipeakon solutions of Novikov's cubically nonlinear integrable Cammasa-Holm type equation.* Dynamics of PDE **6** (2009), no. 3, 253–289.

A. Hone and J. P. Wang, *Integrable peakon equations with cubic nonlinearity.* Preprint (2009).

D. Karapetyan *Non-uniform dependence and well-posedness for the hyperelastic rod equation*, J. Differential Equations **249** (2010) 796–826.

T. Kato and G. Ponce, *Commutator estimates and the Euler and Navier-Stokes equations*. Comm. Pure Appl. Math. **41** (1988), no. 7, pp. 891-907.

C. Kenig, G. Ponce and L. Vega, *On the ill-posedness of some canonical dispersive equations*. Duke Math. J. **106**, no. 3, (2001), 617–633.

References (cont.)

H. Koch and N. Tzvetkov, *Nonlinear wave interactions for the Benjamin-Ono equation*. Int. Math. Res. Not. **30** (2005), 1833-1847.

J. Lenells, *Traveling wave solutions of the Degasperis-Procesi equation*. J. Math. Anal. Appl. **306** (2005), no. 1, 72-82.

Y. Li and P. Olver, *Well-posedness and blow-up solutions for an integrable nonlinearly dispersive model wave equation*. J. Differential Equations **162** (2000), pp. 27-63.

A. Mikhailov and V. Novikov, *Perturbative symmetry approach*. J. Phys. A **35** (2002), no. 22, 47754790.

Z. Qiao, *New integrable hierarchy, its parametric solutions, cuspons, one-peak solitons, and M/W-shape peak solitons.* J. Math. Phys. **48** (2007), no. 8, 08270, 20 pp.

F. Tiglay, *The Periodic Cauchy Problem for Novikov's Equation*. arxiv:1009.1820 (2010).

M. Taylor, *Commutator estimates*. Proc. Amer. Math. Soc. 131 (2003), no. 5, 1501–1507.