MODEL EQUATIONS

$$u_t + (n(u) + Lu)_x = 0$$

- Whitham equation:

$$\widehat{(Lu)}(k) = \underbrace{\left(\frac{\tanh(k)}{k}\right)^{\frac{1}{2}}}_{= m(k)} \hat{u}(k), \qquad n(u) = u^2 + \cdots$$

- KdV equation:

$$Lu = 1 + \tfrac{1}{6}u'', \qquad n(u) = u^2$$

Travelling waves: $u(x,t) = u(x - \nu t)$

Long-wave derivation of KdV from Whitham:

- Long-wave expansion:

$$\nu = 1 + \mu^\gamma \nu_{\mathrm{lw}}, \qquad u(x) = \mu^a w(\mu^\beta x),$$

  where $\mu = \frac{1}{2}\int u^2 := Q(u)$

- Choosing $a = \frac{2}{3}$, $\beta = \frac{1}{3}$, $\gamma = \frac{2}{3}$, we find that

$$\mu^{\frac{4}{3}}\left(\tfrac{1}{6}w'' - \nu_{\mathrm{lw}} w + w^2\right) + o(\mu^{\frac{4}{3}}) = 0$$

KdV has stable solitary-wave solutions. Does Whitham?

VARIATIONAL PRINCIPLES

KdV:

- Solitary waves are local minimisers of

$$\mathcal{E}_{\mathrm{KdV}}(w) := \int \left(\tfrac{1}{12}(w')^2 - \tfrac{1}{3}w^3 \right), \qquad Q(w) = 1$$

- Semilinear structure
- A nonempty set of minimisers over H^1, minimising sequences converge (Albert, Zeng)

Whitham:

- Solitary waves are local minimisers of

$$\mathcal{E}(u) := \int \left(-\tfrac{1}{2}uLu - N(u) \right), \qquad Q(u) = \mu$$

- Linear part is smoothing: $L : H^s \longrightarrow H^{s+\frac{1}{2}}$

$$\|u\|_s^2 := \int (1 + |k|^2)^s \, |\hat{u}(k)|^2 dk$$

- Look for minimisers over

$$U = \{u \in H^1 : \|u\|_1 < R\}$$

- Minimising sequences with $\sup \|u_n\|_1 < R$ converge

PERIODIC WAVES

- Minimise

$$\mathcal{E}_P(u) := \int_{-\frac{P}{2}}^{\frac{P}{2}} \left(-\tfrac{1}{2}uLu - N(u)\right), \qquad Q_P(u) = \mu$$

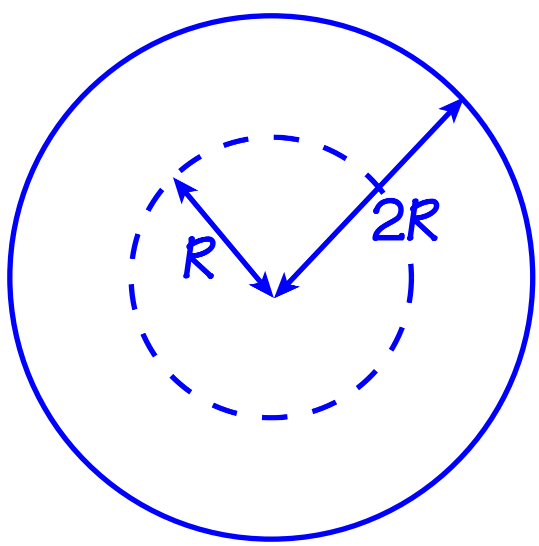
over

$$U_P = \{u \in H^1_P : \|u\|_1 < R\}$$

- We regularise and penalise the functional:

$$\mathcal{E}_{P,\rho}(u) = \mathcal{E}_P(u) + \rho(\|u\|_1^2)$$

  - ρ is smooth and increasing
  - $\rho(t) = 0$ for $0 \le t < R^2$
  - $\rho(t) \to \infty$ as $t \uparrow (2R)^2$



- Look for minimisers of $\mathcal{E}_{P,\rho}$ over $\{\|u\|_1 < 2R\}$

MINIMISATION PROCEDURE

$$\mathcal{E}_{P,\rho}(u) = \underbrace{\mathcal{E}_P(u)}_{\text{Defined on } H^s_P} + \underbrace{\rho(\|\eta\|_1^2)}_{\text{Defined on } H^1_P}, \qquad s \in (\tfrac{1}{2}, 1)$$

- $\mathcal{E}_{P,\rho}$ has a minimiser $u_P \neq 0$, since it is
  - bounded below
  - weakly lower-semicontinuous
  - coercive ($\|u\|_1 \to 2R \quad \Rightarrow \quad \mathcal{E}_{P,\rho}(u) \to \infty$)

  on $\{u \in H^1_P : \mathcal{Q}_P(u) = \mu\}$

- u_P lies in the region unaffected by the penalisation:
  - A priori estimates show that

    $$\mathcal{E}_{P,\rho}(u_P) < -\mu, \qquad \mathcal{E}'_{P,\rho}(u_P) + \nu_P \mathcal{Q}'(u_P) = 0$$

    $$\Rightarrow \quad \nu_P > 1 - \varepsilon, \quad \|u_P\|_1^2 \leq c\mu$$
  - Take $w \in C_0^\infty$ and $\mathcal{Q}(w) = 1$:

    $$\mathcal{E}_P(\mu^{\frac{2}{3}} w(\mu^{\frac{1}{3}} x)) = -\mu + \mu^{\frac{5}{3}} \mathcal{E}_{\mathrm{lw}}(w) + o(\mu^{\frac{5}{3}})$$
  - Take $w(x) = \sqrt{\lambda}\psi(\lambda x)$, $\lambda \ll 1$:

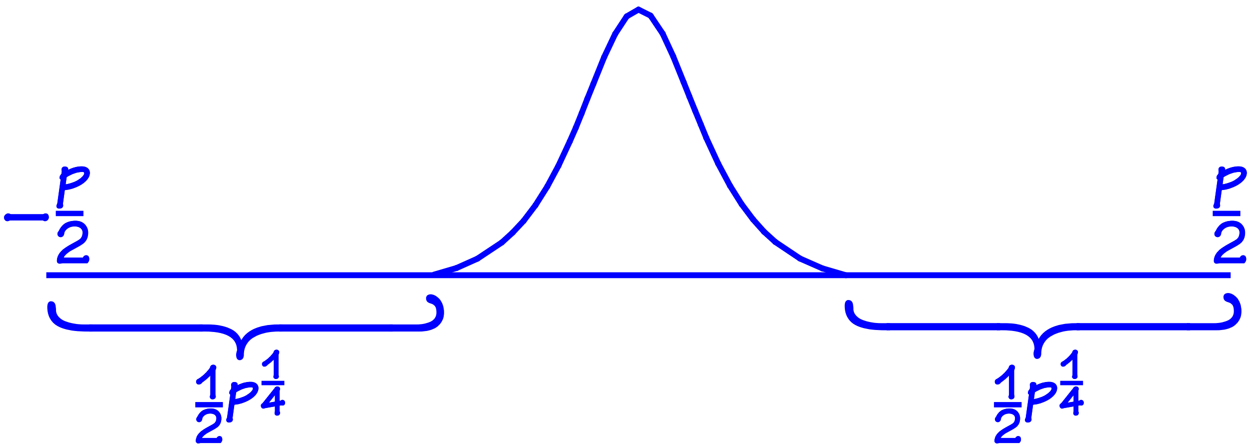
    $$\mathcal{E}_{\mathrm{lw}}(w) = \int \left(\tfrac{1}{12}\lambda^2 (\psi')^2 - \tfrac{1}{3}\lambda^{\frac{1}{2}} \psi^3 \right) < 0$$

SPECIAL MINIMISING SEQUENCE

- Let $\{v_p\}_p$ be a bounded family of functions in H^1 with

$$supp(v_p) \subset (-\tfrac{p}{2}, \tfrac{p}{2}), \quad dist(\pm\tfrac{p}{2}, supp(v_p)) \geq \tfrac{1}{2}p^{\frac{1}{4}}$$

and let $w_p \in H^1_p$ be its periodic extension



- $$\mathcal{E}(v_p) - \mathcal{E}_p(w_p) \longrightarrow 0, \qquad \mathcal{Q}(v_p) = \mathcal{Q}_p(w_p)$$

$$\|\mathcal{E}'(v_p) - \mathcal{E}'_p(w_p)\|_{H^1(-\frac{p}{2},\frac{p}{2})}, \|\mathcal{E}'(v_p)\|_{H^1(\{|x|>\frac{p}{2}\})} \longrightarrow 0$$

$$\|\mathcal{Q}'(v_p) - \mathcal{Q}'_p(w_p)\|_{H^1(-\frac{p}{2},\frac{p}{2})}, \|\mathcal{Q}'(v_p)\|_{H^1(\{|x|>\frac{p}{2}\})} = 0$$

- We translate and truncate u_{p_n} to obtain a minimising squence $\{\tilde{u}_n\}$ for $\mathcal{E}$ with

$$\|\tilde{u}_n\|_1^2 \leq c\mu, \qquad \|\mathcal{E}'(\tilde{u}_n) + \nu_n \mathcal{Q}'(\tilde{u}_n)\|_1 \longrightarrow 0$$

CONCENTRATION-COMPACTNESS

A minimising sequence $\{u_n\}$ with $\mathcal{E}(u_n) \to I_\mu$, $\mathcal{Q}(u_n) = \mu$, $\sup \|u_n\|_1 < R$ can

- concentrate ($\{u_n\}$ behaves like a minimising sequence for the periodic problem)
  – leads to convergence to a minimiser
- vanish (the wave dissolves into ripples)
  – easily ruled out
- dichotomise (the wave splits into two parts)
  - Dichotomy is ruled out by strict sub-additivity

    $$I_{\mu_1+\mu_2} < I_{\mu_1} + I_{\mu_2}$$

    (easy to prove when $\mathcal{N}(u)$ is homogeneous)
  - Try to approximate $\mathcal{N}(u_n)$ with $\frac{1}{3}\int u_n^3$
  - $\mathcal{E}(u_n) < -\mu - c\mu^{\frac{5}{3}} \quad \Rightarrow \quad \mathcal{N}(u_n) \leq -c\mu^{\frac{5}{3}}$
  - So we require $\mathcal{N}_r(u_n) = o(\mu^{\frac{5}{3}})$ (not clear for a general minimising sequence)
  - $\mathcal{N}_r(\tilde{u}_n) = O(\|\tilde{u}_n\|_\infty^2 \|\tilde{u}_n\|_0^2) = O(\|\tilde{u}_n\|_1^4) = O(\mu^2)$

SCALING

- Do solutions KdV scale ($u(x) = \mu^{\frac{2}{3}} w(\mu^{\frac{1}{3}} x)$)?
- Show that $|||u|||^2_{\tau,\mu} \leq c\mu$:
  - $|||v|||^2_{\tau,\mu} := \int \left(v^2 + \mu^{-\frac{4\tau}{3}}(v'')^2\right), \qquad \tau < 1$
  - $||v||_\infty \leq c\mu^{\frac{\tau}{6}} |||v|||_{\tau,\mu}$
- Splitting:

$$\hat{u}_1(k) := \xi(k)\hat{u}(k), \qquad \hat{u}_2(k) := (1 - \xi(k))\hat{u}(k)$$

$$\begin{aligned} (\nu - m)\hat{u}_1 &= \xi \mathcal{F}[n(u)], \\ (\nu - m)\hat{u}_2 &= (1 - \xi)\mathcal{F}[n(u)] \end{aligned}$$

- Estimates:
  - $||u_2''||_0 \leq c||(n(u))''||_0 \leq c||u||_\infty ||u''||_0$
  - $\nu - m(k) > (\nu - 1) + \frac{c}{6}k^2 > \frac{c}{6}k^2$ for $|k| < k_0$, so that

$$\begin{aligned} \int |u_1''|^2 &\leq c \int (\nu - m)^2 |\hat{u}_1|^2 \\ &\leq c||n(u)||_0^2 \\ &\leq c||u||_\infty^2 ||u||_0^2 \end{aligned}$$

SCALING

$$|||v|||^2_{\tau,\mu} := \int \left(v^2 + \mu^{-\frac{4\tau}{3}} (v'')^2 \right)$$

- A supercritical solution satisfies $\|u\|_1^2 \leq c\mu$ and $\|u\|_{k+1} \leq c\|u\|_1$
- We know that

$$\int |u''|^2 \leq c\|u\|_2^2 \|u\|_\infty^2 \leq c\mu^{1+\frac{\tau}{3}} |||u|||^2_{\tau,\mu}$$

- Multiply by $\mu^{-\frac{4\tau}{3}}$, add $\int u^2 = 2\mu$:

$$\mu^{-1} |||u|||^2_{\tau,\mu} \leq c \left(1 + \mu^{1-\tau} (\mu^{-1} |||u|||^2_{\tau,\mu}) \right)$$

- $Q := \{\tau \in [0,1) : \|u\|^2_{\tau,\mu} \leq c\mu\}$
  - $0 \in Q$
  - $\tau \in Q \implies [0,\tau] \subset Q$
  - Suppose $\tau_\star := \sup Q < 1$. Choose $\varepsilon > 0$ so that $\tau_\star + \frac{11}{3}\varepsilon < 1$:

$$\mu^{-1} |||u|||^2_{\tau_\star+\varepsilon,\mu} \leq c \left(1 + \mu^{1-\tau_\star-\frac{11}{3}\varepsilon} \underbrace{(\mu^{-1} |||u|||^2_{\tau_\star-\varepsilon,\mu})}_{\leq c} \right)$$

CONVERGENCE

- $$\sup_{u \in D_\mu} \mathrm{dist}_{H^1}(\mu^{-\frac{2}{3}} u(\mu^{-\frac{1}{3}} x), D_{\mathrm{lw}}) \longrightarrow 0$$

  Otherwise:

  - There exist $\{\mu_n\}$ and $\{u_n\} \in D_{\mu_n}$ with

  $$\mu_n \longrightarrow 0,$$

  $$\inf_{w \in D_{\mathrm{lw}}} \|\mu^{-\frac{2}{3}} u_n(\mu^{-\frac{1}{3}} x) - w\|_1 \geq \varepsilon$$

  - $\{\mu_n^{-\frac{2}{3}} u_n(\mu_n^{-\frac{1}{3}} x)\}$ is a minimising sequence for $\mathcal{E}_{\mathrm{lw}}$ over $\{w \in H^1 : Q(w) = 1\}$, so converges

- There exists a family $\{w_u\}_{u \in D_\mu}$ of functions in D_{lw} such that

  $$\nu(u) = 1 + \mu^{\frac{2}{3}} \nu_{\mathrm{lw}}(w_u) + o(\mu^{\frac{2}{3}})$$

  uniformly over $u \in D_\mu$.