Splash Singularities for the Water Waves Problem

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Joint work with D. Córdoba, C. Fefferman
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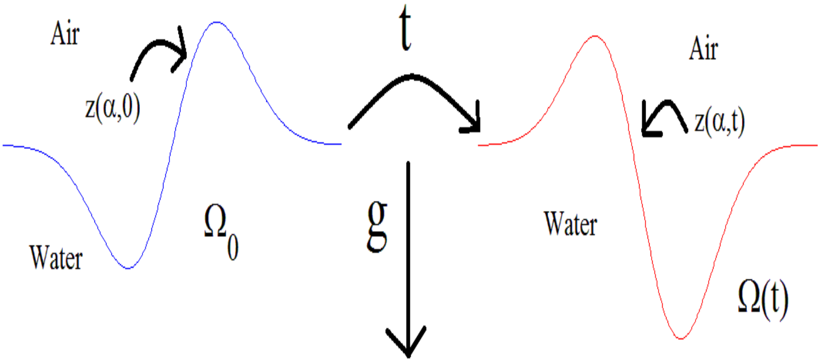
The Water Wave problem



Physical laws

- The incompressibility condition

$$\nabla \cdot u = 0$$

- The mass conservation equation

$$\rho_t + u \cdot \nabla \rho = 0$$

$$\rho(x_1, x_2, t) = \begin{cases} 0, & x \in \mathbb{R}^2 \setminus \Omega(t) \\ \rho = 1, & x \in \Omega(t) \end{cases}$$

- Euler equation

$$\rho(u_t + (u \cdot \nabla)u) = -\nabla p - (0, \mathrm{g}\rho)$$

- Irrotational fluid

$$\nabla \times u = 0$$

Irrotational flows

We assume that the vorticity is zero in the interior of the domain $\Omega(t)$. We can consider that the vorticity is supported on the free boundary curve $z(\alpha, t)$ and it has the form

$$\omega(x,t) = \varpi(\alpha,t)\delta(x - z(\alpha,t)).$$

i.e. the vorticity is a Dirac measure on z defined by

$$\langle \nabla^{\perp} \cdot u, \eta \rangle = \int_{\mathbb{R}} \varpi(\alpha,t)\eta(z(\alpha,t))d\alpha,$$

with $\eta(x)$ a test function.

Scenario

We consider:

1. Asymptotically flat curves

$$\lim_{\alpha \to \infty} (z(\alpha, t) - (\alpha, 0)) = 0.$$

2. Periodic curves in the horizontal variable

$$z(\alpha + 2k\pi, t) = z(\alpha, t) + 2k\pi(1, 0).$$

3. Closed contours

$$z(\alpha + 2k\pi, t) = z(\alpha, t).$$

The Contour Equations

The equation for the curve

$$z_t(\alpha,t) \;=\; BR(z,\varpi)(\alpha,t)+c(\alpha,t)\partial_\alpha z(\alpha,t),$$

The equation for the amplitude of the vorticity

$$\begin{aligned}\varpi_t(\alpha,t) \;=\; & -2\partial_t BR(z,\varpi)\cdot\partial_\alpha z-\partial_\alpha\Big(\frac{|\varpi|^2}{4|\partial_\alpha z|^2}\Big)+\partial_\alpha(c\,\varpi)\\ & +2c\,\partial_\alpha BR(z,\varpi)\cdot\partial_\alpha z(\alpha,t)-2\mathrm{g}\partial_\alpha z_2.\end{aligned}$$

Here

$$BR(z,\varpi)=\frac{1}{2\pi}\int\frac{(z(\alpha,t)-z(\beta,t))^{\perp}}{|z(\alpha,t)-z(\beta,t)|^2}\varpi(\beta,t)d\beta.$$

Rayleigh-Taylor condition

A linearization around a flat contour $(\alpha, f(\alpha,t))$, allows us to find

$$f_t = \frac{1}{2} H\omega$$
$$\omega_t = -\sigma \partial_\alpha f$$

- Rayleigh-Taylor condition:

$$\sigma(\alpha,t) = -(\nabla p^2(z(\alpha,t),t) - \nabla p^1(z(\alpha,t),t)) \cdot \partial_\alpha^\perp z(\alpha,t) > 0,$$

Existence of solutions

- Local Existence

Theorem (Sijue Wu, 1997)

Local existence for initial data satisfying $z_0(\alpha) \in H^5$ *and* $\varpi_0(\alpha) \in H^4$,

$$\mathcal{F}(z_0)(\alpha, \beta) < \infty, \quad \textit{and} \quad \sigma(\alpha, 0) > 0.$$

Here

$$\mathcal{F}(z)(\alpha, \beta, t) = \frac{|\beta|}{|z(\alpha, t) - z(\alpha - \beta, t)|} \qquad \forall\, \alpha, \beta \in (-\pi, \pi),$$

and

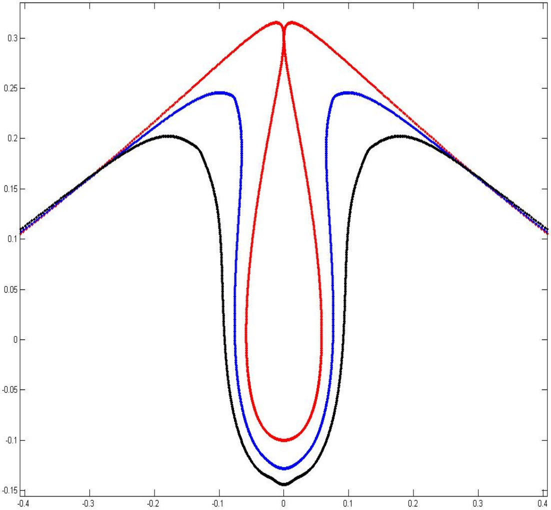
$$\mathcal{F}(z)(\alpha, 0, t) = \frac{1}{|\partial_\alpha z(\alpha, t)|}.$$

- Global Existence
  - In 3D, global exsistence for small initial data has been proven by S. Wu and also by P. Germain N. Masmoudi and J. Shatah.
  - In 2D, almost global existence for small initial has been proven by S. Wu.

References

- Nalimov (1974).
- Yoshihara (1982).
- Craig (1985).
- Beale- Hou-Lowengrub (1993).
- Christodoulou-Lindblad (2000).
- Lindblad (2005).
- Lannes (2005).
- Ambrose-Masmoudi (2005).
- Coutand-Shkoller (2007).
- Zhang-Zhang (2008).
- Alazard-Metivier (2009).
- Alazard-Burq-Zuily (2009).
- Córdoba-Córdoba-Gancedo (2010).
- ....

Splash singularity



Singularities for Water waves: Theorems

Joint work with D. Córdoba, C. Fefferman, F. Gancedo and J. Gómez-Serrano.

Theorem (Existence)

There exists a smooth solution of water waves problem which develops a splash singularity in finite time.

Theorem (Stability)

Close to a quasi-solution which start in a Splash curve there is a exact solution which start in the same splash curve.

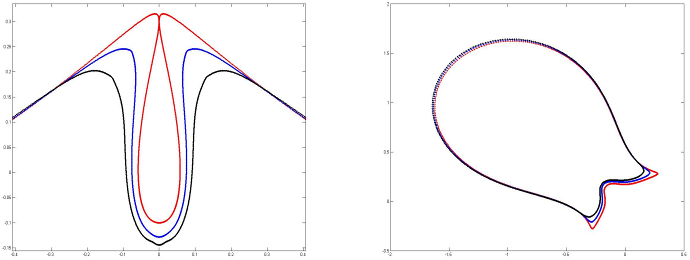
References

A. C., D. Córdoba, C. Fefferman, F. Gancedo and J. Gómez-Serrano. Splash singularties for the water waves. Proc. Natl. Acad. Sci. 109 (3):733-738 (2012).

A. C., D. Córdoba, C. Fefferman, F. Gancedo and J. Gómez-Serrano. Finite time singularities for the free boundary Euler equation. Arxiv preprint. arXiv:1112.2170.

A. C., D. Córdoba, C. Fefferman, F. Gancedo and J. Gómez-Serrano. Finite time singularities for water waves with surface tension. Arxiv preprint. arXiv:1204.6633v1

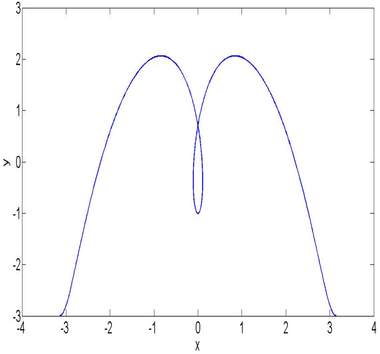
Steps of the proof of splash singularity

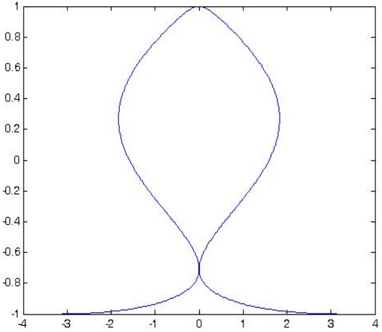


$$P(w) = \left(\tan\left(\frac{w}{2}\right)\right)^{1/2}, \quad w \in \mathbb{C},$$

- The water wave equations are invariant under time reversal. To obtain a solution that ends in a splash, we can therefore take our initial condition to be a splash, and show that there is a smooth solution for small times $t > 0$.

Non-Splash Curves





The water waves equations

Let assume that $z(\alpha, t)$ is smooth and satisfies the chord-arc condition. We have to solve

$$\left.\begin{array}{c}\nabla\cdot u=0\\ \nabla\times u=0\end{array}\right\}\text{ in }\Omega(t)$$

We can write

$$u=\nabla\phi \qquad u=\nabla^{\perp}\psi$$

with

$$\begin{array}{cc}\Delta\phi=0 & \Delta\psi=0\\ \phi|_S=\Phi & \partial_n\psi|_S=-\frac{\partial_\alpha\Phi(\alpha)}{|z(\alpha,t)|}\end{array}$$

We can use potential theory to write

$$u(z(\alpha,t),t)\cdot z_\alpha(\alpha,t)=\partial_\alpha\Phi(\alpha,t)=BR(z,\varpi)\cdot z_\alpha(\alpha,t)+\frac{\varpi(\alpha,t)}{2}$$

for some function $\varpi(\alpha,t)$ (the amplitude of the vorticity).
Also we have to solve

$$\partial_t u+(u\cdot\nabla)u=-\nabla p-(0,1)\quad \text{in } \Omega(t)$$

that we can write

$$\partial_t\phi+\frac{1}{2}|u|^2=-p-y.$$

We will take

$$p|_{\text{surf}}=0$$

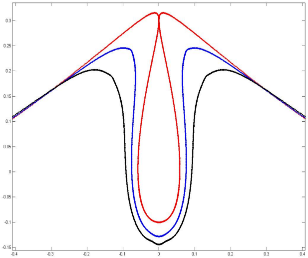
Since the interface is transported by the flow we have that

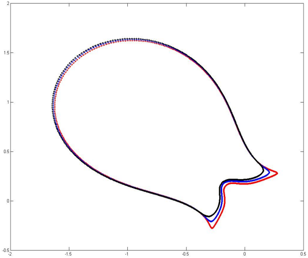
$$z_t = u(z(\alpha,t),t) + c(\alpha,t)z_\alpha(\alpha,t)$$

And we can check that in the interfaces we have the equation

$$\begin{aligned}
\partial_t z(\alpha,t) =& BR(z,\omega)(\alpha,t) + \overline{c}(\alpha,t)\partial_\alpha z(\alpha,t)\\
\partial_t \omega(\alpha,t) =& -2\partial_\alpha z(\alpha,t)\cdot\partial_t BR(z,\omega)(\alpha,t)\\
&-\partial_\alpha\left(\frac{|\omega|^2}{4|\partial_\alpha z|^2}\right)(\alpha,t) + \partial_\alpha\left(\overline{c}(\alpha,t)\omega(\alpha,t)\right)\\
&+2\overline{c}(\alpha,t)\partial_\alpha z(\alpha,t)\cdot\partial_\alpha BR(z,\omega)(\alpha,t) - 2g\partial_\alpha z_2(\alpha,t).
\end{aligned}$$

For a splash curve we cannot use the amplitude of the vorticity.





$$\Omega(t) \qquad \longrightarrow \qquad \tilde{\Omega}(t)$$

$$P(w) = \left(\tan\left(\frac{w}{2}\right)\right)^{\frac{1}{2}}$$

We define the new potencial

$$\tilde{\psi}(\tilde{x},\tilde{y},t) \equiv \psi(P^{-1}(\tilde{x},\tilde{y}),t), \qquad \tilde{\phi}(\tilde{x},\tilde{y},t) \equiv \phi(P^{-1}(\tilde{x},\tilde{y}),t)$$

the new velocity,

$$\tilde{v}(\tilde{x},\tilde{y},t) \equiv \nabla\tilde{\phi}(\tilde{x},\tilde{y},t)$$

and the restriction

$$\tilde{\Phi}(\alpha,t) = \tilde{\phi}(\tilde{z}(\alpha,t),t), \qquad \tilde{\Psi}(\alpha,t) = \tilde{\psi}(\tilde{z}(\alpha,t),t).$$

Thus

$$\begin{aligned} \Delta\tilde{\phi}(\tilde{x},\tilde{y},t) &= 0 \quad \text{in } P(\Omega(t)) \\ \tilde{\phi}\Big|_{\tilde{z}(\alpha,t)} &= \tilde{\Phi}(\alpha,t) \\ \tilde{v} \equiv \nabla\tilde{\phi} & \quad \text{in } P(\Omega(t)) \end{aligned}$$

And also

$$\Phi(\alpha,t) = \tilde{\Phi}(\alpha,t) \quad \Psi(\alpha,t) = \tilde{\Psi}(\alpha,t) \quad u_{\text{normal}}(\alpha,t) = \tilde{u}_{\text{normal}}(\alpha,t)$$

IMPORTANT: In $P(\Omega(t)) = \tilde{\Omega}(t)$ we can use the amplitude of the vorticity $\tilde{w}$ even if the curve in $\Omega(t)$ is a splash curve.

From the equations

$$\begin{aligned} z_t =& u(\alpha, t) + c(\alpha, t) z_\alpha \\ \Phi_t =& \phi_t + z_t \cdot u \\ =& \frac{1}{2} u(\alpha, t)^2 + c(\alpha, t) u(\alpha, t) - g z^2(\alpha, t) \end{aligned}$$

(notice there is no ϖ anywhere) we obtain the equations in tilda domain.

- Equations in the new domain

$$\tilde{z}_t(\alpha,t) = Q^2(\alpha,t)BR(\tilde{z},\tilde{\omega})(\alpha,t) + \tilde{c}(\alpha,t)\tilde{z}_\alpha(\alpha,t)$$

$$\tilde{\omega}_t(\alpha,t) = -2\partial_t BR(\tilde{z},\tilde{\omega})(\alpha,t)\cdot\tilde{z}_\alpha(\alpha,t) - |BR(\tilde{z},\tilde{\omega})|^2\partial_\alpha Q^2(\alpha,t)$$

$$+ 2\tilde{c}(\alpha,t)\partial_\alpha BR(\tilde{z},\tilde{\omega})\cdot\tilde{z}_\alpha(\alpha,t) + \partial_\alpha\left(\tilde{c}(\alpha,t)\tilde{\omega}(\alpha,t)\right) - 2\partial_\alpha\left(P_2^{-1}(\tilde{z}(\alpha,t))\right).$$

where

$$Q^2(\alpha,t) = \left|\frac{dP}{dw}(z(\alpha,t))\right|^2.$$

Local Existence in the Tilde Domain

Theorem

Let $z^0(\alpha)$ *be a splash curve. Let* $u^0(\alpha)\cdot(z^0_\alpha)^\perp(\alpha)\in H^4(\mathbb{T})$ *satisfying:*

1. $u^0(\alpha_1)\cdot\frac{(z^0_\alpha)^\perp}{|z^0_\alpha|}<0, u^0(\alpha_2)\cdot\frac{(z^0_\alpha)^\perp}{|z^0_\alpha|}<0.$
2. $\int_{\partial\Omega}u^0\cdot\frac{(z^0_\alpha)^\perp}{|z^0_\alpha|}ds=\int_{\mathbb{T}}u^0(\alpha)\cdot(z^0_\alpha)^\perp d\alpha=0.$

Then there exist a finite time $T>0$*, a curve* $\tilde{z}(\alpha,t)\in C([0,T];H^4)$ *satisfying:*

1. $P^{-1}(\tilde{z}_1(\alpha,t))-\alpha, P^{-1}(\tilde{z}_2(\alpha,t))$ *are* 2π*-periodic,*
2. $P^{-1}(\tilde{z}(\alpha,t))$ *satisfies the arc-chord condition for all* $t\in(0,T]$*,*

and $\tilde{u}(\alpha,t)\in C([0,T];H^3(\mathbb{T}))$ *which provides a solution of the water waves equations in the new domain* $\tilde{z}^0(\alpha)=P(z^0(\alpha))$*.*

The proof of this theorem is based on

A. Córdoba, D. Córdoba and F. Gancedo. Interface evolution: water wave in 2-D. Adv. Math., 223, no. 1, 120-173 (2009).

A priori energy estimates

$$E(t) = \|\tilde{z}\|^2_{H^3}(t) + \int_{\mathbb{T}} \frac{Q^2\sigma_{\tilde{z}}}{|\tilde{z}_\alpha|^2}|\partial^4_\alpha \tilde{z}|^2 d\alpha(t) + \|F(\tilde{z})\|^2_{L^\infty}(t)$$
$$+ \|\tilde{\omega}\|^2_{H^2}(t) + \|\varphi\|^2_{H^{3+\frac{1}{2}}}(t) + \frac{|\tilde{z}_\alpha|^2}{m(Q^2\sigma_{\tilde{z}})(t)} + \sum_{l=0}^{4} \frac{1}{m(q_l)(t)}$$

where

$$\varphi(\alpha,t) = \frac{Q^2(\alpha,t)\tilde{\omega}(\alpha,t)}{2|\tilde{z}_\alpha(\alpha,t)|} - \tilde{c}(\alpha,t)|\tilde{z}_\alpha(\alpha,t)|$$
$$c(\alpha,t) = \frac{\alpha+\pi}{2\pi}\int_{-\pi}^{\pi}(Q^2BR(\tilde{z},\tilde{\omega}))_\beta(\beta,t)\cdot\frac{\tilde{z}_\beta(\beta,t)}{|\tilde{z}_\beta(\beta,t)|^2}d\beta$$
$$-\int_{-\pi}^{\alpha}(Q^2BR(\tilde{z},\tilde{\omega}))_\beta(\beta,t)\cdot\frac{\tilde{z}_\beta(\beta,t)}{|\tilde{z}_\beta(\beta,t)|^2}d\beta$$

$\sigma_{\tilde{z}}$ is the R-T function.

Singular points of Q

Singular points

$$q^0 = (0,0)\,, \quad q^1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right),$$
$$q^2 = \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \quad q^3 = \left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right),$$
$$q^4 = \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right).$$

The function φ allows two show the following cancelation:

$$\frac{d}{dt}\int_{\mathbb{T}}\frac{Q^2\sigma_{\tilde{z}}}{|\tilde{z}_\alpha|^2}|\partial_\alpha^4\tilde{z}|^2 d\alpha = \text{Controlled Quantities} + S$$
$$\frac{d}{dt}\int \Lambda\partial_\alpha^3\varphi(\alpha,t)\partial_\alpha\varphi(\alpha,t)d\alpha = \text{Controlled Quantities} - S$$

where

$$S = \int_{-\pi}^{\pi} 2Q^2\sigma\frac{\partial_\alpha^4 z\cdot z_\alpha^\perp}{|z_\alpha|^3}\Lambda(\partial_\alpha^3\varphi)d\alpha,$$

The R-T function

We define $\tilde{p}(\tilde{x},\tilde{y},t) = p(P^{-1}(\tilde{x},\tilde{y}),t)$. Then

$$\sigma_{\tilde{z}} = -(\nabla \tilde{p})(\tilde{z},t)\cdot \tilde{z}_\alpha^\perp = \sigma_z$$

and one can check that

$$\begin{aligned}\sigma_{\tilde{z}} = &\left(BR_t(\tilde{z},\tilde{\omega}) + \frac{\varphi}{|\tilde{z}_\alpha|}BR_\alpha(\tilde{z},\tilde{\omega})\right)\cdot \tilde{z}_\alpha^\perp + \frac{\tilde{\omega}}{2|\tilde{z}_\alpha|^2}\left(\tilde{z}_{\alpha t} + \frac{\varphi}{|\tilde{z}_\alpha|}\tilde{z}_{\alpha\alpha}\right)\cdot \tilde{z}_\alpha^\perp \\ &+ Q\left|BR(\tilde{z},\tilde{\omega}) + \frac{\tilde{\omega}}{2|\tilde{z}_\alpha|^2}\tilde{z}_\alpha\right|^2 (\nabla Q)(\tilde{z})\cdot \tilde{z}_\alpha^\perp - (\nabla P_2^{-1})(\tilde{z})\cdot \tilde{z}_\alpha^\perp\end{aligned}$$

We have that

$$\begin{aligned}-\Delta p =& |\nabla u|^2 > 0 \quad u \neq 0 \\ -p - gy =& O(1)\end{aligned}$$

By applying Hopf's lemma one can check that $\sigma_{\tilde{z}} > 0$.

The regularization

$$z_t^{\varepsilon,\delta,\mu}(\alpha,t) = \phi_\delta * \phi_\delta * \left(Q^2(z^{\varepsilon,\delta,\mu}) BR(z^{\varepsilon,\delta,\mu}, \omega^{\varepsilon,\delta,\mu})\right)(\alpha,t)$$
$$+ \phi_\mu * \left(c^{\varepsilon,\delta,\mu}\left(\phi_\mu * \partial_\alpha z^{\varepsilon,\delta,\mu}\right)\right)(\alpha,t),$$

$$\begin{aligned}\omega_t^{\varepsilon,\delta,\mu} &= + \ldots \\ &+ \ldots \\ &- 2\varepsilon \frac{|\partial_\alpha z^{\varepsilon,\delta,\mu}|}{Q^2(z^{\varepsilon,\delta,\mu})} \Lambda(\phi_\mu * \phi_\mu * \varphi^{\varepsilon,\delta,\mu}),\end{aligned}$$

$$
\begin{aligned}
c^{\varepsilon,\delta,\mu}(\alpha) =& \frac{\alpha+\pi}{2\pi}\int_{-\pi}^{\pi} \frac{\partial_\beta z^{\varepsilon,\delta,\mu}(\beta))}{|\partial_\beta z^{\varepsilon,\delta,\mu}(\beta)|^2} \\
& \times \phi_\delta * \phi_\delta * (\partial_\beta(Q^2(z^{\varepsilon,\delta,\mu})(\beta)BR(z^{\varepsilon,\delta,\mu},\omega^{\varepsilon,\delta,\mu}))(\beta))d\beta \\
& - \int_{-\pi}^{\alpha} \frac{\partial_\beta z^{\varepsilon,\delta,\mu}(\beta)}{|\partial_\beta z^{\varepsilon,\delta,\mu}(\beta)|^2} \\
& \times \phi_\delta * \phi_\delta * (\partial_\beta(Q^2(z^{\varepsilon,\delta,\mu})(\beta)BR(z^{\varepsilon,\delta,\mu},\omega^{\varepsilon,\delta,\mu}))(\beta))d\beta,
\end{aligned}
$$

$$
\varphi^{\varepsilon,\delta,\mu} = \frac{Q^2(z^{\varepsilon,\delta,\mu})\omega^{\varepsilon,\delta,\mu}}{2|\partial_\alpha z^{\varepsilon,\delta,\mu}|} - \mathcal{C}^{\varepsilon,\delta,\mu},
$$

$$
\begin{aligned}
\mathcal{C}^{\varepsilon,\delta,\mu} =& \phi_\delta * \phi_\delta * \left(\frac{\alpha+\pi}{2\pi}\int_{-\pi}^{\pi} \frac{\partial_\beta z^{\varepsilon,\delta,\mu}(\beta)}{|\partial_\beta z^{\varepsilon,\delta,\mu}(\beta)|} \cdot (\partial_\beta(Q^2(z^{\varepsilon,\delta,\mu})BR(z^{\varepsilon,\delta,\mu},\omega^{\varepsilon,\delta,\mu})))(\beta)d\beta\right) \\
& - \phi_\delta * \phi_\delta * \left(\int_{-\pi}^{\alpha} \frac{\partial_\beta z^{\varepsilon,\delta,\mu}(\beta)}{|\partial_\beta z^{\varepsilon,\delta,\mu}(\beta)|} \cdot (\partial_\beta(Q^2(z^{\varepsilon,\delta,\mu})(\beta)BR(z^{\varepsilon,\delta,\mu},\omega^{\varepsilon,\delta,\mu}))(\beta))d\beta\right).
\end{aligned}
$$

Stability

(x, γ, ζ) are the solutions of

$$\begin{cases} x_t & = Q^2(x)BR(x,\gamma) + bx_\alpha + f \\ b & = \underbrace{\frac{\alpha+\pi}{2\pi}\int_{-\pi}^{\pi}(Q^2BR)_\alpha \frac{x_\alpha}{|x_\alpha|^2}d\alpha - \int_{-\pi}^{\alpha}(Q^2BR)_\beta \frac{x_\alpha}{|x_\alpha|^2}d\beta}_{b_s} \\ & + \underbrace{\frac{\alpha+\pi}{2\pi}\int_{-\pi}^{\pi} f_\alpha \frac{x_\alpha}{|x_\alpha|^2}d\alpha - \int_{-\pi}^{\alpha} f_\beta \frac{x_\beta}{|x_\beta|^2}d\beta}_{b_e} \\ \gamma_t & +2BR_t(x,\gamma)\cdot x_\alpha = -(Q^2(x))_\alpha |BR(x,\gamma)|^2 + 2bBR_\alpha(x,\gamma)\cdot x_\alpha \\ & \qquad +(b\gamma)_\alpha - \left(\frac{Q^2(x)\gamma^2}{4|x_\alpha|^2}\right)_\alpha - 2(P_2^{-1}(x))_\alpha + g \\ \zeta(\alpha,t) & = \frac{Q_x^2(\alpha,t)\gamma(\alpha,t)}{2|x_\alpha(\alpha,t)|} - b_s(\alpha,t)|x_\alpha(\alpha,t)| \end{cases}$$

$$\begin{aligned} D(\alpha,t) &\equiv z(\alpha,t) - x(\alpha,t) \\ d(\alpha,t) &\equiv \omega(\alpha,t) - \gamma(\alpha,t) \\ \mathcal{D}(\alpha,t) &\equiv \varphi(\alpha,t) - \zeta(\alpha,t) \end{aligned}$$

$$\mathcal{E}(t) \equiv \left(\|D\|_{H^3}^2 + \int_{-\pi}^{\pi} \frac{Q^2\sigma_z}{|z_\alpha|^2} |\partial_\alpha^4 D|^2 + \|d\|_{H^2}^2 + \|\mathcal{D}\|_{H^{3+\frac{1}{2}}}^2 \right).$$

Then we have that

$$\left| \frac{d}{dt}\mathcal{E}(t) \right| \leq \mathcal{C}(t)(\mathcal{E}(t) + \delta(t))$$

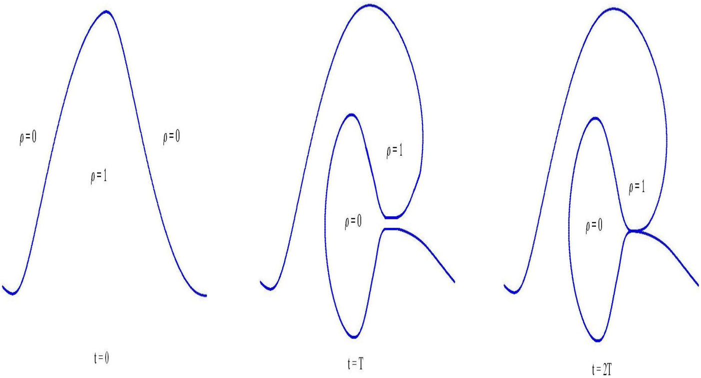
where

$$\mathcal{C}(t) = \mathcal{C}(E(t), \|x\|_{H^{5+\frac{1}{2}}}(t), \|\gamma\|_{H^{3+\frac{1}{2}}}(t), \|\zeta\|_{H^{4+\frac{1}{2}}}(t), \|F(x)\|_{L^\infty}(t))$$

and

$$\delta(t) = (\|f\|_{H^{5+\frac{1}{2}}}(t) + \|g\|_{H^{3+\frac{1}{2}}}(t))^k$$

Thing to do



Thing to do

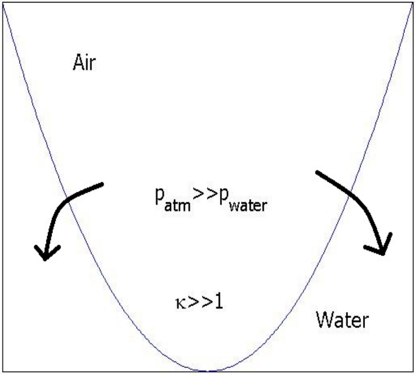
- Numerical analysis gives a simulation which start being a graph and ends in a splash curve.
- Using this simulation we can construct a curve $x(\alpha, t)$ and an amplitude $\gamma(\alpha, t)$ (just by interpolating in space and time).
- This couple, $x(\alpha, t)$ and $\gamma(\alpha, t)$, is not a solution but it is a quasi-solution.
- Using the computer we can estimate the errors f and g. We hope these errors to be small.
- We need to compute the constant of the stability theorem.
- Finally, we apply the stability theorem.

Splash singularities with surface tension

Laplace-Young law

$$p_{\text{atm}} - p_{\text{fluid}} = \tau\kappa$$

where τ is a constant which depend on the fluid we are considering and κ is the curvature.



Water Waves equations with surface tension

In the physical domain

$$\begin{aligned} z_t(\alpha,t) &= BR(z,\varpi)(\alpha,t) + c(\alpha,t)\partial_\alpha z(\alpha,t), \\ \varpi_t(\alpha,t) &= -2\partial_t BR(z,\varpi)\cdot\partial_\alpha z - \partial_\alpha\Big(\frac{|\varpi|^2}{4|\partial_\alpha z|^2}\Big) + \partial_\alpha(c\,\varpi) \\ &\quad + 2c\,\partial_\alpha BR(z,\varpi)\cdot\partial_\alpha z(\alpha,t) - 2\mathrm{g}\partial_\alpha z_2 + \tau\partial_\alpha\kappa(\alpha,t) \end{aligned}$$

In the tilda domain

$$\begin{aligned} \tilde{z}_t(\alpha,t) =& Q^2(\alpha,t)BR(\tilde{z},\tilde{\omega})(\alpha,t) + \tilde{c}(\alpha,t)\tilde{z}_\alpha(\alpha,t) \\ \tilde{\omega}_t(\alpha,t) =& -2\partial_t BR(\tilde{z},\tilde{\omega})(\alpha,t)\cdot\tilde{z}_\alpha(\alpha,t) - |BR(\tilde{z},\tilde{\omega})|^2\partial_\alpha Q^2(\alpha,t) \\ & - \partial_\alpha\left(\frac{Q^2(\alpha,t)}{4}\frac{\tilde{\omega}(\alpha,t)^2}{|z_\alpha(\alpha,t)|^2}\right) + 2\tilde{c}(\alpha,t)\partial_\alpha BR(\tilde{z},\tilde{\omega})\cdot\tilde{z}_\alpha(\alpha,t) \\ & + \partial_\alpha\left(\tilde{c}(\alpha,t)\tilde{\omega}(\alpha,t)\right) - 2\partial_\alpha\left(P_2^{-1}(\tilde{z}(\alpha,t))\right) + \tau NT(\alpha,t). \end{aligned}$$

The new term

$$NT(\alpha,t) = \partial_\alpha\left(Q\tilde{\kappa}(\alpha,t)\right) + \text{extra lower terms}$$

The new energy

- Energy without the R-T condition.

$$E = \ldots + 2|\tilde{z}_\alpha|^3 \int Q^7 \left(\partial_\alpha^3 \tilde{\kappa}\right)^2 + \frac{1}{\tau} \int Q^8 \partial_\alpha^3 \tilde{\varpi} \Lambda \partial_\alpha^3 \tilde{\varpi} + \frac{1}{2|\tilde{z}_\alpha|\tau^2} \int Q^9 \left(\partial_\alpha^3 \tilde{\varpi}\right)^2 \tilde{\varpi}^2$$

Ambrose (2003)

- Energy with the R-T condition.

$$E = \ldots + \frac{\tau|\tilde{z}_\alpha|^3}{2} \int Q^7 \left(\partial_\alpha^3 \tilde{\kappa}\right)^2 + \int Q^4 \partial_\alpha^3 \tilde{\varphi} \Lambda \tilde{\varphi} + |\tilde{z}_\alpha|^2 \tau \int \left(C||\tilde{\kappa}||_{H^1}(t) + \tilde{\kappa}\right) Q^7 \partial_\alpha^2 \tilde{\kappa} \Lambda \partial_\alpha^2 \tilde{\kappa} + 2|\tilde{z}_\alpha| C||\tilde{\kappa}||_{H^1}(t) \int Q^4 \left(\partial_\alpha^3 \tilde{\varpi}\right)^2 + |\tilde{z}_\alpha| \int \sigma Q^6 \left(\partial_\alpha^3 \tilde{\kappa}\right)^2$$

Ambrose and Masmoudi (2005) and Ambrose and Masmoudi (2009)

THANK YOU FOR YOUR ATTENTION