On the Cauchy problem for water gravity waves

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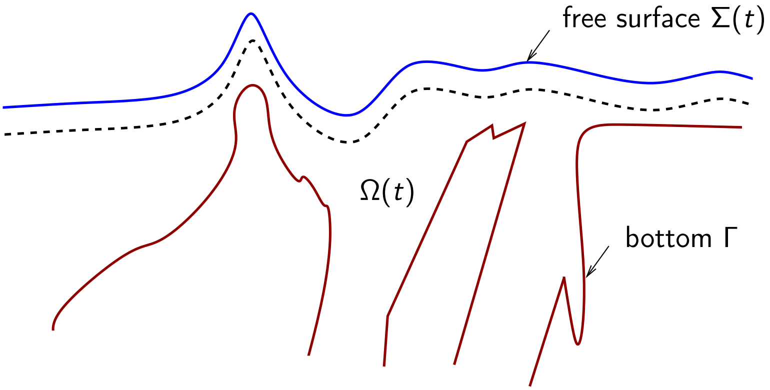
Nonlinear waves and interface problems

Study the dynamics of an incompressible, irrotational liquid flow

- moving under the force of gravitation,
- in a time-dependent domain Ω with a free boundary Σ.



The fluid domain

The spatial coordinates are (x, y), $x \in \mathbb{R}^d$, $y \in \mathbb{R}$, $d \geq 1$.

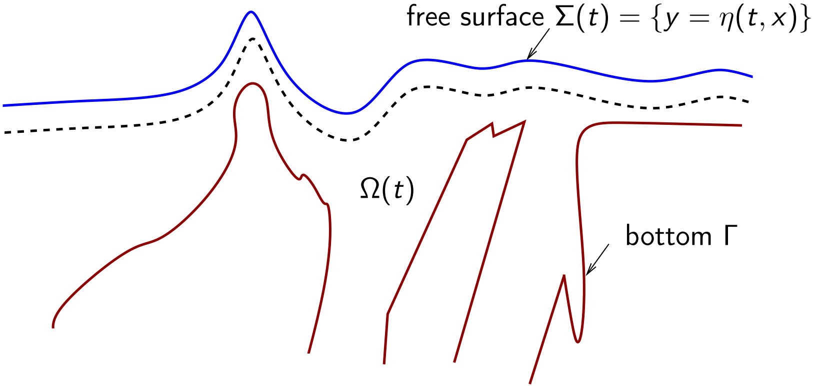
Let $\mathcal{O} \subset \mathbb{R}^d \times \mathbb{R}$, $d \geq 1$, be an open connected domain. Set

$$\Omega(t) = \{(x, y) \in \mathcal{O} \, : \, y < \eta(t, x)\} \, ,$$
$$\Sigma(t) = \{(x, y) \in \mathcal{O} \, : \, y = \eta(t, x)\} \, ,$$
$$\Gamma = \partial \mathcal{O}$$

where η is an unknown. Assumption :

$$\exists h > 0 \text{ s.t. } \Big\{(x, y) \in \mathbb{R}^d \times \mathbb{R} \, : \, \eta(t, x) - h < y < \eta(t, x)\Big\} \subset \Omega(t).$$

Examples: $\mathcal{O} = \mathbb{R}^d \times \mathbb{R}$ or $\mathcal{O} = \mathbb{R}^d \times [-1, +\infty)$.



$\Sigma(t)$ is a graph and $\Omega(t)$ is connected

$1 \lesssim \mathrm{dist}(\Sigma(t), \Gamma)$

no assumption on $\Gamma = \partial\mathcal{O}$

no restriction on $d \geq 1$

The equations

We consider an incompressible inviscid liquid, having unit density.

The velocity field $v\colon \Omega \to \mathbb{R}^{d+1}$ solves the incompressible Euler equation

$$\partial_t v + v \cdot \nabla_{x,y} v + \nabla_{x,y}(P + gy) = 0, \quad \operatorname{div}_{x,y} v = 0 \quad \text{in } \Omega,$$

where $g > 0$ is the acceleration of gravity, P is the pressure. In addition

$$\operatorname{curl}_{x,y} v = 0 \quad \text{in } \Omega.$$

The problem is then given by three boundary conditions:

$$\begin{aligned}
&\text{on the bottom } \Gamma && v \cdot n = 0, \\
&\text{on the free surface } \Sigma && \partial_t \eta = \sqrt{1 + |\nabla \eta|^2}\; v \cdot n\,, \qquad (\star) \\
&\text{on the free surface } \Sigma && P = 0.
\end{aligned}$$

Linearized equation. Laplace, Lagrange, Cauchy and Poisson

Since $\operatorname{curl}_{y,x} v = 0$ and since $\operatorname{div}_{x,y} v = 0$,

$$v = \nabla_{x,y}\phi \quad \text{with } \Delta_{x,y}\phi = 0 \quad \text{in } \Omega.$$

Expanding the solution in terms of a small parameter ε :

$$\eta = \varepsilon\eta_1 + \varepsilon^2\eta_2 + \cdots, \quad \phi = \varepsilon\phi_1 + \varepsilon^2\phi_2 + \cdots$$

and removing $O(\varepsilon^2)$, we find

$$\begin{cases} \partial_t\eta - |D_x|\,\psi = 0, \quad \text{where } \psi(t,x) = \phi(t,x,0), \\ \partial_t\psi + g\eta = 0. \end{cases}$$

Hence $u = |D_x|^{\frac{1}{2}}\,\psi + i\sqrt{g}\eta$, $u = u(t,x) \in \mathbb{C}$, satisfies

$$\partial_t u + i\sqrt{g}\,|D_x|^{\frac{1}{2}}\,u = 0.$$

The solution of $\partial_t u + i\,|D_x|^{1/2}\,u = 0$ with initial data u_0 is given by

$$u(t) = S(t)u_0, \qquad S(t) = \exp(-it\,|D_x|^{1/2}).$$

$S(t)$ is not bounded on $L^p(\mathbb{R}^d)$ for $p \neq 2$, $t \neq 0$, $d \geq 1$ (Fefferman–Stein).

We thus work in $H^s(\mathbb{R}^d)$.

Consider the case $d = 2$.

For all $\epsilon > 0$, there exists $C > 0$ such that

Strichartz $\quad \|u\|_{L^2_t((0,1);L^\infty_x(\mathbb{R}^2))} \leq C\|u_0\|_{H^{\frac{3}{4}+\epsilon}(\mathbb{R}^2)},$

Sobolev $\quad \|u\|_{L^\infty_t((0,1);L^\infty_x(\mathbb{R}^2))} \leq C\|u_0\|_{H^{1+\epsilon}(\mathbb{R}^2)}.$

Energy / Scaling

Assume that $v = v(t,x,y)$ solves the Euler's equations in $\{y < \eta(t,x)\}$.

The energy is conserved:

$$\frac{d}{dt}\left\{\frac{1}{2}\int_{\Omega(t)} |v(t,x,y)|^2 \, dxdy + \frac{g}{2}\int_{\mathbb{R}^d} \eta(t,x)^2 \, dx\right\} = 0.$$

We do not know if weak solutions exist at this level of regularity.

No further coercive conservation laws (P. Olver).

Energy / Scaling

v_λ and η_λ defined by

$$v_\lambda(t,x,y) = \lambda^{-1/2} v(\sqrt{\lambda}t, \lambda x, \lambda y), \quad \eta_\lambda(t,x) = \lambda^{-1}\eta(\sqrt{\lambda}t, \lambda x),$$

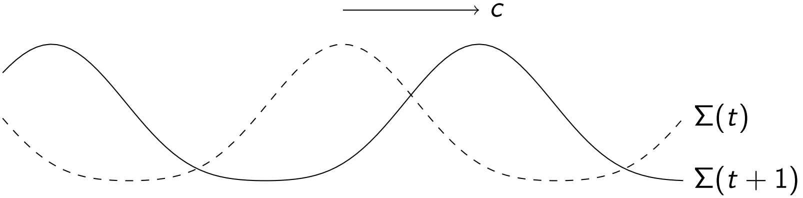
solve the same equations.

Then the **critical case** corresponds to $\nabla \eta_0 \in L^\infty(\mathbb{R}^d)$ since

$$\|\nabla \eta_0\|_{L^\infty} = \left\|\lambda^{-1}\nabla(\eta_0(\lambda\cdot))\right\|_{L^\infty}.$$

Stokes waves of extremal form

Consider $2D$-progressive waves (that is $d = 1$):



Lewy : $C^1 \Rightarrow C^\omega$.

Stokes, Plotnikov, Amick–Fraenkel–Toland : there exist solutions such that

$$\eta(t,x) = \widetilde{\eta}(x - ct), \quad v(t,x,y) = \widetilde{v}(x - ct, y),$$

with $\widetilde{\eta}$ Lipschitz but not C^1.

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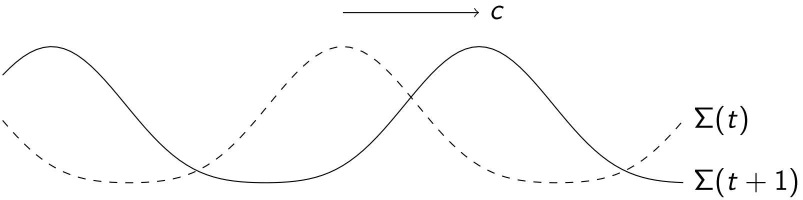

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Pionneering works: Lagrange, Laplace, Poisson, Cauchy, Stokes, Boussinesq. Nalimov, Ovsjannikov, Yosihara, Craig, Shinbrot, Beale, T. Hou & J. Lowengrub, Guido Schneider, C. Eugene Wayne

Wu, Beyer-Günther

Two different approaches were used: the Lagrangean formulation with a geometrical point of view and the Eulerian formulation with ΨDO analysis.

Geometric analysis: Christodoulou–Lindblad, Lindblad, Shatah–Zeng, Coutand–Shkoller, Ambrose-Masmoudi, Keel–Zhao; Castro–Córdoba–Fefferman–Gancedo–Gómez Serrano

Singular integrals analysis: Here we follow the approach initiated by Craig–Schanz–Sulem and further developed by Lannes;

Para-differential analysis: A.-Métivier (inspired by works by Alinhac);

Dispersive analysis for the equation with surface tension: Christianson-Hur-Staffilani, Spirn-Wright; A.–Burq–Zuily, Chen-Marzuola-Spirn-Wright;

Small solutions : Wu, Germain–Masmoudi–Shatah, Iooss–Plotnikov, Buffoni–Groves–Sun–Wahlén,

Regularity thresholds

A) Without viscosity : lipshitz regularity threshold for the velocity.

B) In terms of the free boundary, there is no such natural criteria.

Christodoulou–Lindblad : The Sobolev norms remain bounded as long as the second fundamental form of the free surface is bounded, and the first-order derivatives of the velocity are bounded (non irrotationnal case, general free surface).

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Let $\eta_0 \in H^{s+\frac{1}{2}}(\mathbb{R}^d)$ and $v_0|_{y=\eta_0} \in H^s(\mathbb{R}^d)$. If

$$s < \frac{1}{2} + \frac{d}{2} \Rightarrow \text{" Ill-posedness"}; \quad s > \frac{3}{2} + \frac{d}{2} \Rightarrow \textit{a priori} \text{ bounds.}$$

Well-posedness for larger s : Wu, Linblad, Lannes, Shatah–Zeng, Coutand–Shkoller, Córdoba-Córdoba-Gancedo, Masmoudi-Rousset.

$$s > \frac{3}{2} + \frac{d}{2} \Rightarrow \textit{a priori} \text{ bounds}, \quad s \text{ large} \;\Rightarrow\; \text{LWP}$$

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- For $s > 1 + \frac{d}{2} - \delta$ with $\delta > 0$ small enough, *a priori* estimates:

$$\|u\|_{L^\infty([0,T];H^s(\mathbb{R}^2))} + \|u\|_{L^2([0,T];W^{1+\varepsilon,\infty}(\mathbb{R}^2))} \le C(\|u_0\|_{H^s(\mathbb{R}^2)}),$$

  for some $\varepsilon > 0$.
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- For $s > \frac{3}{4} + \frac{d}{2}$, reduction of the water waves system to a wave-type equation on the boundary.
- For $s > \frac{1}{2} + \frac{d}{2}$, one can 1) define and estimate all the terms (including the pressure); 2) prove that the Taylor sign condition is automatically satisfied in infinite depth; 3) pass to the limits in the equations.

Main result

Theorem (A.-Burq-Zuily)

Assume that initially $1 \lesssim \mathrm{dist}(\Sigma, \Gamma)$ *and* $1 \lesssim -\partial_y P|_\Sigma$ *(automatic if* $\Gamma = \emptyset$*).*

(i) *Denote by* $\underline{v} = v|_\Sigma$ *the trace of the velocity field on the free surface. The Cauchy problem is (locally in time) well-posed for initial data* $(\eta_0, \underline{v}_0) \in H^{s+\frac{1}{2}} \times H^s$ *for* $s > 1 + d/2$*.*

(ii) *For* $d = 2$, a priori *estimates and existence hold for* $s > 1 + d/2 - 1/12$*.*

(i) : The initial velocity is lipschitz and η is $C_x^{3/2}$;

(ii) : The initial velocity is not lipschitz but

$$\|\underline{v}\|_{L^2(0,T;W^{1,\infty}(\mathbb{R}^d))} < +\infty.$$

We study a nonlinear system

- whose coefficients are given by solving elliptic equations in domain with rough boundaries,
- whose linearized system around the 0 solution can be written as

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We diagonalize the nonlinear equations, by introducing a good unknown u which solves a dispersive equation of the form

$$\partial_t u + T_V \cdot \nabla u + i\,T_\gamma u = f,$$

V is the trace of the horizontal component of the velocity on Σ;

$\gamma = \gamma(t, x, \xi)$ is a symbol of order $1/2$;

T_V and T_γ are paradifferential operators.

For 2D waves, one can further reduce matters to

$$\partial_t u + T_W \partial_x u + i\,|D_x|^{1/2}\,u = f.$$

First main tool : Dirichlet–Neumann operator

Aim : reduction to the boundary (Craig–Sulem's formulation).

Given two functions $\eta = \eta(x)$ and $f = f(x)$, define $G(\eta)f$ by

$$G(\eta)f = \partial_y \phi - \nabla\eta \cdot \nabla\phi \Big|_{y=\eta} = \sqrt{1+|\nabla\eta|^2}\,\partial_n\phi \Big|_{y=\eta}.$$

where $\phi = \phi(x,y)$ is given by

$$\Delta_{x,y}\phi = 0 \quad \text{in } \Omega, \qquad \phi\Big|_{\Sigma} = f, \quad \partial_n\phi = 0 \quad \text{on } \Gamma.$$

$G(\eta)$ is the Dirichlet–Neumann operator.

Many results : Craig-Schanz-Sulem, Wu, Beyer-Günther, Lannes

Proposition

Consider a domain with an arbitrary bottom such that the distance between the free surface and the bottom is bounded from below by a fixed positive constant h.

- *If* $\eta \in W^{1,\infty}(\mathbb{R}^d)$ *and* $f \in H^{1/2}(\mathbb{R}^d)$*, then one can define a unique variational solution* ϕ *to*

$$\Delta_{x,y}\phi = 0 \quad \text{in } \Omega, \qquad \phi\big|_{y=\eta} = f, \quad \partial_n \phi|_\Gamma = 0.$$

- *Let* $s > 1/2 + d/2$ *(scaling). If* $(\eta, f) \in H^{s+\frac{1}{2}}(\mathbb{R}^d) \times H^s(\mathbb{R}^d)$*, then*

$$\|G(\eta)f\|_{H^{s-1}} \le C\left(\|\eta\|_{H^{s+\frac{1}{2}}}\right)\|f\|_{H^s}.$$

- *Let* $s > 1 + d/2$*. Contraction :*

$$\|[G(\eta_1) - G(\eta_2)]\, f\|_{H^{s-\frac{3}{2}}} \le C\left(\|(\eta_1,\eta_2)\|_{H^{s+\frac{1}{2}}}, \|f\|_{H^s}\right)\|\eta_1 - \eta_2\|_{H^{s-\frac{1}{2}}}.$$

Since we consider low regularity solutions, several terms are not well-defined. Consequently, various cancellations have to be used.

Since we consider low regularity solutions, several terms are not well-defined. Consequently, various cancellations have to be used. Set

$$\zeta = \nabla \eta, \quad V = v_{\mathrm{horizontal}}|_{\Sigma}, \quad B = v_{\mathrm{vertical}}|_{\Sigma}, \quad \mathfrak{a} = -\partial_y P|_{\Sigma}.$$

Proposition

There holds

$$\begin{aligned} &(\partial_t + V \cdot \nabla) B = \mathfrak{a} - g, \\ &(\partial_t + V \cdot \nabla) V + \mathfrak{a}\zeta = 0, \\ &(\partial_t + V \cdot \nabla)\zeta = G(\eta)V + \zeta G(\eta)B. \end{aligned}$$

Study progressive waves and the Cauchy problem in the same setting.

Various cancellations (for instance $G(\eta)B = -\operatorname{div} V$).

Paralinearization and symmetrization

By using the Fourier transform : $G(0) \simeq |D_x|$.

If $\eta \in C^\infty$, it is known since Calderón that $G(\eta)$ is a ΨDO of order 1, whose principal symbol is

$$\lambda(x,\xi) := \sqrt{(1+|\nabla\eta(x)|^2)\,|\xi|^2 - (\nabla\eta(x)\cdot\xi)^2}.$$

More precisely,

$$G(\eta)f = (2\pi)^{-d}\int e^{ix\cdot\xi}\lambda(x,\xi)\widehat{u}(\xi)\,d\xi \;+\; R_0(\eta)f,$$

where the remainder satisfies

$$\exists K \geq 1, \forall s \geq 0, \quad \|R_0(\eta)\psi\|_{H^s} \leq C\left(\|\eta\|_{H^{s+K}}\right)\|\psi\|_{H^s}.$$

Remark: If $d=1$ or $\eta=0$ then $\lambda(x,\xi)=|\xi|$ and $\mathrm{Op}(\lambda)=|D_x|$.

Paraproducts:

$$\begin{aligned} ab &= \iint_{\mathbb{R}^d \times \mathbb{R}^d} e^{2i\pi x\cdot(\theta+\xi)}\, \widehat{a}(\theta)\, \widehat{b}(\xi)\, d\theta d\xi \\ &= \iint_{|\theta+\xi|\sim|\xi|} + \iint_{|\theta+\xi|\sim|\theta|} + \iint_{|\theta|\sim|\xi|} \\ &= T_a b + T_b a + R(a,b) \end{aligned}$$

$T_a b$ has the same regularity as b, and $R(a,b)$ is twice more regular.

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Given a function $a = a(x)$, define the paraproduct operator T_a by

$$
\widehat{T_a u}(\xi) = (2\pi)^{-d}\int \chi(\xi-\eta,\eta)\widehat{a}(\xi-\eta)\widehat{u}(\eta)\,d\eta,
$$

where $\widehat{a}(\theta) = \int e^{-ix\cdot\theta}a(x)\,dx$ and

$$
\begin{aligned}
&\chi(\theta,\eta) = 1 \quad \text{for} \quad |\theta| \le \varepsilon_1 |\eta|, \\
&\chi(\theta,\eta) = 0 \quad \text{for} \quad |\theta| \ge \varepsilon_2 |\eta|, \quad 0<\varepsilon_1<\varepsilon_2\ll 1.
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Given a symbol $a = a(x,\eta)$, define the paradifferential operator T_a by

$$\widehat{T_a u}(\xi) = (2\pi)^{-d} \int \chi(\xi-\eta,\eta)\widehat{a}(\xi-\eta,\eta)\widehat{u}(\eta)\,d\eta,$$

where $\widehat{a}(\theta,\eta) = \int e^{-ix\cdot\theta} a(x,\eta)\,dx$ and

$$\begin{aligned} \chi(\theta,\eta) &= 1 \quad \text{for} \quad |\theta| \leq \varepsilon_1 |\eta|, \\ \chi(\theta,\eta) &= 0 \quad \text{for} \quad |\theta| \geq \varepsilon_2 |\eta|, \quad 0 < \varepsilon_1 < \varepsilon_2 \ll 1. \end{aligned}$$

Definition

Let $m \in \mathbb{R}$, $\rho \in \mathbb{R}$ and consider $a = a(x, \xi)$.
$a \in \Gamma^m_\rho$ iff a is homogeneous of order m in ξ, with regularity C^ρ in x :

$$\left\| \partial_\xi^\alpha a(\cdot, \xi) \right\|_{C^\rho} \leq K_\alpha (1 + |\xi|)^{m - |\alpha|}, \quad \forall \alpha \in \mathbb{N}^d, \forall |\xi| \geq 1.$$

For $\rho \geq 0$, $C^\rho = W^{\rho,\infty}$. For $\rho < 0$, C^ρ is the Zygmund space of index ρ.

Example: $\eta \in C^1$ implies $\lambda \in \Gamma^1_0$.

Theorem (Bony)

Let $m, m' \in \mathbb{R}$ and $\rho \in [0, 1]$.

(i) If $a \in \Gamma^m_0$, then T_a is of order m (bounded from H^s to H^{s-m}, $\forall s \in \mathbb{R}$).

(ii) If $a \in \Gamma^m_\rho, b \in \Gamma^{m'}_\rho$. Then

$$T_a T_b \text{ and } T_{ab} \text{ are of order } m + m',$$

$$T_a T_b - T_{ab} \text{ is of order } m + m' - \rho.$$

λ is well-defined for any $\eta \in C^1$. Compare $G(\eta)$ to T_λ when $\eta \in C^\rho$, $1 \le \rho \le 2$.

If $s > 2 + d/2$ then $R(\eta) = G(\eta) - T_\lambda$ satisfies

$$\left\| R(\eta) f \right\|_{H^s} \le C \left(\left\| \eta \right\|_{H^{s+1}} \right) \left\| f \right\|_{H^s} ,$$

However, when f is only one-half derivative less regular than η,

$$\left\| R(\eta) f \right\|_{H^{s-\frac{1}{2}}} \le C \left(\left\| \eta \right\|_{H^{s+\frac{1}{2}}} \right) \left\| f \right\|_{H^s} .$$

The regularity of the remainder term is given by the regularity of the domain.

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Proposition (Up to the scaling)

If $s > 1/2 + d/2$ then

$$\|R(\eta)f\|_{H^{s-1+\varepsilon}} \leq C\left(\|\eta\|_{H^{s+\frac{1}{2}}}\right)\|f\|_{H^s} \quad \varepsilon = \min\left\{\frac{1}{2}, s - \frac{1}{2} - \frac{d}{2}\right\},$$

hence, if $s > 1 + d/2$ then

$$\|R(\eta)f\|_{H^{s-1/2}} \leq C\left(\|\eta\|_{H^{s+\frac{1}{2}}}\right)\|f\|_{H^s}.$$

Paralinearization of the DNO

Theorem (Tame estimate)

Let $d \geq 1$, *and*

$$s > \frac{3}{4} + \frac{d}{2}, \qquad r > 1.$$

Then $R(\eta)f = G(\eta)f - T_\lambda f$ *satisfies*

$$\|R(\eta)f\|_{H^{s-\frac{1}{2}}} \leq C\left(\|\eta\|_{H^{s+\frac{1}{2}}}, \|f\|_{H^s}\right)\left\{1 + \|\eta\|_{C^{\frac{3}{2}}} + \|f\|_{C^r}\right\}.$$

To compare $\|\cdot\|_{H^{s+\frac{1}{2}}}$ and $\|\cdot\|_{C^{3/2}}$, for $s < 1 + d/2$, we notice

$$\left\|u\left(\frac{x}{\varepsilon}\right)\right\|_{C^{\frac{3}{2}}} \sim \left(\frac{1}{\varepsilon}\right)^{3/2} >> \left(\frac{1}{\varepsilon}\right)^{s+\frac{1}{2}-\frac{d}{2}} \sim \left\|u\left(\frac{x}{\varepsilon}\right)\right\|_{H^{s+\frac{1}{2}}}.$$

Reduction

Recall $\zeta = \nabla\eta, \quad V = v_{\text{horizontal}}|_\Sigma, \quad B = v_{\text{vertical}}|_\Sigma, \quad a = -\partial_y P|_\Sigma.$

The scalar complex-valued unknown u defined by

$$u = T_{\sqrt{a}}\zeta + iT_{\sqrt{\lambda}}(V + T_\zeta B),$$

satisfies

$$\partial_t u + T_V \cdot \nabla u + i\, T_\gamma u = F,$$

where

$$\gamma(t,x,\xi) = \sqrt{a(t,x)\lambda(t,x,\xi)},$$

and F is well estimated: for any $s > 3/4 + d/2$,

$$\|F\|_{L^p(0,T;H^{s-\frac{1}{2}})} \leq C\Big(\|\eta\|_{L^\infty(0,T;H^{s+\frac{1}{2}})}, \|(V,B)\|_{L^\infty(0,T;,H^s)} \Big) \times \Big\{ 1 + \|\eta\|_{L^p(0,T;C^{3/2})} + \|(V,B)\|_{L^p(0,T;C^{1+\varepsilon})} \Big\}.$$

- $\partial_t + V \cdot \nabla$ has the same weight as $|D_x|^{1/2}$.
- For any $s > 3/4 + d/2$, the Taylor's coefficient satisfies

$$\|a(t) - g\|_{H^{s-\frac{1}{2}}} + \|a(t)\|_{C^{\frac{1}{2}}} + \|(\partial_t a + V \cdot \nabla a)(t)\|_{L^\infty}$$
$$\leq C\left(\|\eta(t)\|_{H^{s+\frac{1}{2}}}, \|(V,B)(t)\|_{H^s}\right)\left\{1 + \|\eta(t)\|_{C^{\frac{3}{2}}} + \|(V,B)(t)\|_{C^r}\right\}.$$

- There holds

$$\left\|\left[T_p, \partial_t + T_V \cdot \nabla\right]\right\|_{L^2 \to L^2}$$
$$\leq \|p\|_{L^\infty} \|V\|_{C^{1+\varepsilon}} + \|\partial_t p + V \cdot \nabla p\|_{L^\infty} \|V\|_{L^\infty},$$

Strichartz estimates

Let $d = 2$ and recall that, for all $\epsilon > 0$, there exists $C > 0$ such that,

$$\| e^{it|D_x|^{1/2}} u_0 \|_{L^2_t((0,1);L^\infty_x(\mathbb{R}^2))} \leq C \| u_0 \|_{H^{\frac{3}{4}+\epsilon}(\mathbb{R}^2)}.$$

We want to prove a similar estimate for the equation

$$\partial_t u + T_V \cdot \nabla u + iT_\gamma u = 0.$$

Difficulties:

- Non smooth coefficients. We consider the case where

$$V \in L^2([0,T]; W^{1,\infty}(\mathbb{R}^2)), \qquad \gamma(\cdot,\cdot,\xi) \in L^2([0,T]; C^{1/2}(\mathbb{R}^2)).$$

- Dispersion due to the subprincipal term of order $1/2$, T_γ.
- The equation is pseudo-differential.

- Make a frequency analysis: $u = \sum_{j=-1}^{\infty} \Delta_j u$

$$(\partial_t + S_j(V)\cdot\nabla + iT_\gamma)\Delta_j u = F_j.$$

- Regularize V and γ (inspired by works of Lebeau, Smith, Bahouri-Chemin,Tataru.) Take $0<\delta<1$ (here $\delta=\frac{2}{3}$) and set

$$L_\delta = \partial_t + S_{j\delta}(V)\cdot\nabla + iT_{\gamma_\delta}, \quad \gamma_\delta = \psi(2^{-j\delta}D_x)\gamma$$

$$\text{then} \qquad L_\delta \Delta_j u = G_j$$

- Straighten the vector field $\partial_t + S_{j\delta}(V)\cdot\nabla$

$$\dot{X}(t) = S_{j\delta}(t, X(t)), \quad X(0) = x.$$

Lemma

$$\| \frac{\partial X}{\partial x}(t,\cdot) - Id \|_{L^\infty(\mathbb{R}^d)} \le C|t|^{\frac{1}{2}}$$

$$\|(\partial_x^\alpha X)(t,\cdot)\|_{L^\infty(\mathbb{R}^d)} \le C_\alpha h^{-\delta(|\alpha|-1)}|t|^{\frac{1}{2}}, \quad |\alpha|\ge 2, \quad h = 2^{-j}$$

- Semi-classical approach: we set

$$v_h(t,y) = (\Delta_j u)(t, X(t,y)), \quad h = 2^{-j}.$$

Make the change of variables

$$z = h^{-\frac{1}{2}} y, \quad \tilde{h} = h^{\frac{1}{2}}, \quad w_{\tilde{h}}(t,z) = v_h(t, \tilde{h} y).$$

Then,

$$(\tilde{h}\partial_t + iP(t, \tilde{h}z, \tilde{h}D_z, \tilde{h}))w_{\tilde{h}} = \tilde{h}F_{\tilde{h}}.$$

for a rather explicit operator P of order $1/2$.

We look for a parametrix on a time interval of size $h^\rho = \tilde{h}^{2\rho}$ (where $\rho = \frac{1}{3}$) of the form

$$\mathcal{K}v(t,z) = (2\pi\tilde{h})^{-d} \iint e^{\frac{i}{\tilde{h}}(\phi(t,z,\xi,\tilde{h}) - z'\cdot\xi)} b(t,z,\xi,\tilde{h}) v(z')\, dz'\, d\xi,$$

where b is a symbol and ϕ a real valued phase such that

$$\phi|_{t=0} = z\cdot\xi, \quad b|_{t=0} = \chi(\xi), \quad \mathrm{supp}\chi \subset \{\xi : \frac{1}{3} \le |\xi| \le 3\}.$$

Semi-classical estimates

Using the parametrix, the stationnary phase estimate and coming back to the original variable $z \to y = h^{\frac{1}{2}} z \to x = X(t,y)$ we prove

Proposition

Let $t_0 \in \mathbb{R}$ and consider $u_{0,h} \in L^1(\mathbb{R}^d)$ localized at frequency h.

There exist $C > 0$ and $h_0 > 0$ such that, if U_h solves

$$\Big(\partial_t + \frac{1}{2}(T_{V_\delta} \cdot \nabla + \nabla \cdot T_{V_\delta}) + i T_{\gamma_\delta}\Big) U_h = 0, \quad U_h(t_0, x) = u_{0,h}(x).$$

then

$$\| U_h(t) \|_{L^\infty(\mathbb{R}^d)} \leq C h^{-\frac{3d}{4}} |t - t_0|^{-\frac{d}{2}} \| u_{0,h} \|_{L^1(\mathbb{R}^d)},$$

for all $h \in]0, h_0]$ and $t \in]t_0, t_0 + h^{\frac{1}{3}}]$.

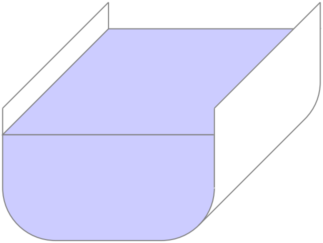
Semi-classical Strichartz estimates on time intervals tailored to the frequency

Lebeau, Smith, Tataru, Staffilani–Tataru, Burq-Gérard-Tzvetkov, Blair

Water waves in a channel

Water waves in a channel

3D waves in a non rectangular channel with vertical walls near the free surface:



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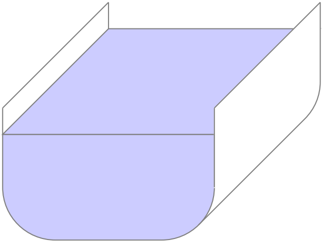

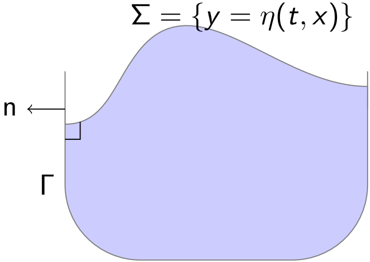

To see the right-angles :

$$\left.\begin{aligned}&\left[\partial_t v + (v\cdot\nabla_{x,y})v\right]\cdot n = 0\\ &e_y\cdot n(x,y) = 0\end{aligned}\right\} \quad\Rightarrow\quad -\nabla_{x,y}P\cdot n(x,y) = 0.$$

Since $P\,|_{\Sigma} = 0$, this implies that $\nabla_{x,y}P$ is proportional to the normal to Σ.

The proof consists in extending the initial data to functions defined on the whole space $\mathbb{R}^2$. We proceed by a symmetry and a periodization (following Boussinesq, 1910).

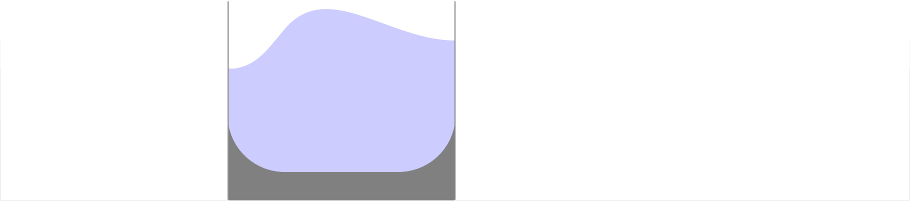

Figure: 2D section of the channel

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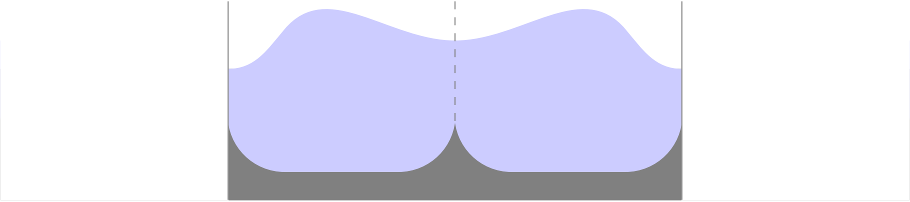

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Rough bottom, even though initially the bottom is smooth.

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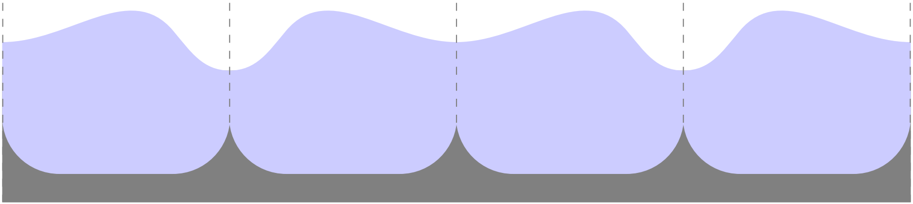

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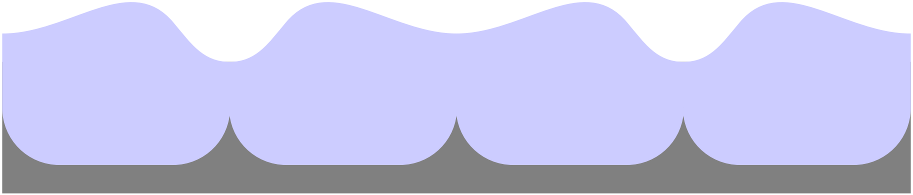

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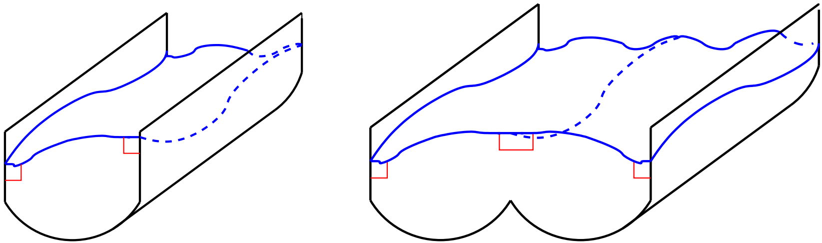
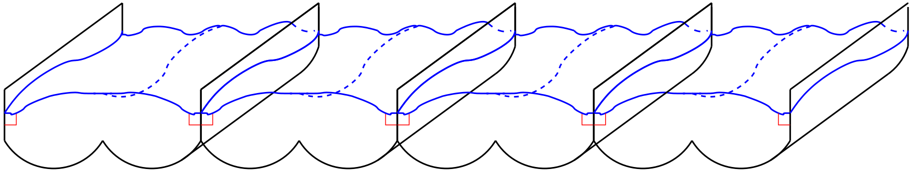


For $s < 3$, if $\partial_{x_1}\eta(0, x_2) = 0 = \partial_{x_1}\eta(1, x_2)$ then

$$\eta \in H^{s+\frac{1}{2}}\big((0,1)_{x_1} \times \mathbb{R}_{x_2}\big) \ \Rightarrow\ \underline{\eta} \in H^{s+\frac{1}{2}}\big(\mathbb{T} \times \mathbb{R}_{x_2}\big).$$

Then we apply our Cauchy theorem to these extended initial data.

Uniqueness for low regularity solutions $\Rightarrow$ the symmetries are propagated
$\Rightarrow$ η is even in x_1 and v_1 is odd in x_1.
$\Rightarrow$ $v \cdot n = 0$ on the wall of the channels.



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