Algebraic Attacks in Post–Quantum Cryptography

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Overview

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Polynomial Systems

Consider a system composed of m quadratic polynomials in n variables over a finite field $\mathbb{F}$. By *solving* we mean finding one common root to these m polynomials.

$$\begin{aligned} p_1(x_1, \ldots, x_n) &= \sum \alpha_{i,j}^{(1)} x_i x_j + \sum \beta_{i,j}^{(1)} x_i + \gamma^{(1)} = 0 \\ \vdots & \qquad\qquad \vdots \\ p_m(x_1, \ldots, x_n) &= \sum \alpha_{i,j}^{(m)} x_i x_j + \sum \beta_{i,j}^{(m)} x_i + \gamma^{(m)} = 0 \end{aligned}$$

Example: Multivariate Public Key Cryptosystems

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  to recover plaintext $(m_1, \ldots, m_n)$.
- **features**: efficient encryption, compact ciphertext, reasonably large public keys, decryption efficiency depends on how easy is to solve the system above (with knowledge of the secret key), post-quantum security.

Linearisation

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Big Field Multivariate Schemes

Idea: let $\mathbb{F}_{2^n}$ be an extension field of $\mathbb{F}_2$ of degree n; then use a polynomial $F \in \mathbb{F}_{2^n}[X]/\langle X^{2^n} + X\rangle$ which is efficiently computable, and efficiently invertible, but try to *hide* its structure by representing the encryption by polynomials $p_1, \ldots, p_n$ in $\mathbb{F}_2[x_1, x_2, \ldots, x_n]$ (using the public isomorphism ϕ).

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Fact: The degree of the polynomials $p_1, \ldots, p_n$ is equal to the maximal Hamming weight of the univariate polynomial. Example $F(X) = X^3 = X^{2^0+2^1} \Rightarrow p_1, \ldots, p_n$ are quadratic.

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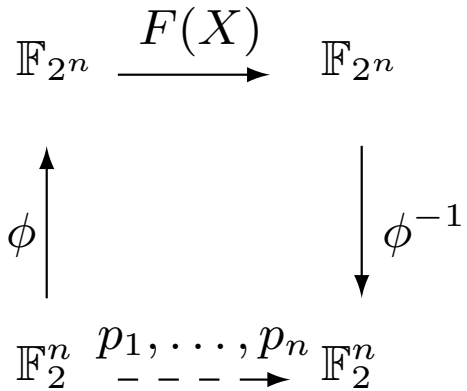

Example: C^*

- **Efficiency**: we may use $F(X) = X^{1+2^\Theta}$. Then inversion can be done efficiently; moreover polynomials $p_1, \ldots, p_n$ are quadratic (since X^{2^Θ} is linear over $\mathbb{F}_2$).

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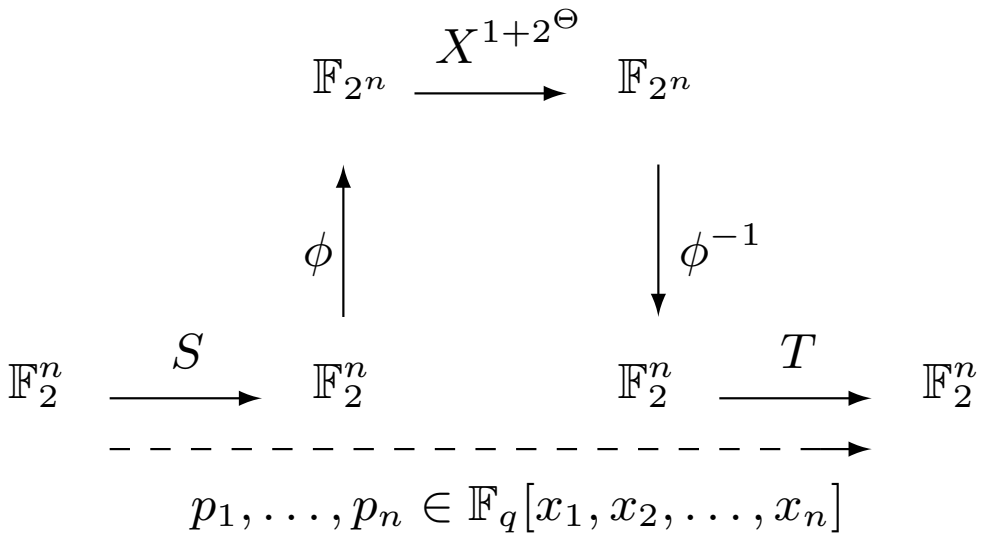

A Weakness of Big–Field Schemes

- Big–Field schemes tend to carry a lot of algebraic structure (hailing from the extension field). These can be exploited by Gröbner basis algorithms.

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- Current iterations of Big–Field Schemes rely on the use of certain modifiers to hide the algebraic structure. Examples include removing public polynomials, or adding extra information (variables or whole polynomials).

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Overview, Big–Field Schemes

There are several variants of secure Big–Field multivariate *signature* schemes:

- GeMMS (variant of HFE$v-$) (Casanova et al., Round 2 NIST Submission, 2017)
- PFLASH (Chen et al., 2015)

[b] Currently analysing security in a joint work with H.Raddum, P. Felke, and C.Cid.

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- EFC (Szepieniec et al., PQC'16): **one variant broken** (Wang et al., 2019)[b]
- HFERP (Ikematsu et al., PQC'18)
- Two–Face (Macario–Rat and Patarin, Africacrypt'18)[b]
- EFLASH (Cartor and Smith–Tone, SAC'18): **broken** (Øygarden et al., CT–RSA'20)

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Algebraically Simple Block Ciphers

- Recent years have seen the design of symmetric ciphers that are optimized for advanced cryptographic protocols such as Multi–Party Computation, Zero–Knowledge proofs and Fully Homomorphic Encryption. A design goal is often to minimize the multiplicative complexity (MC).

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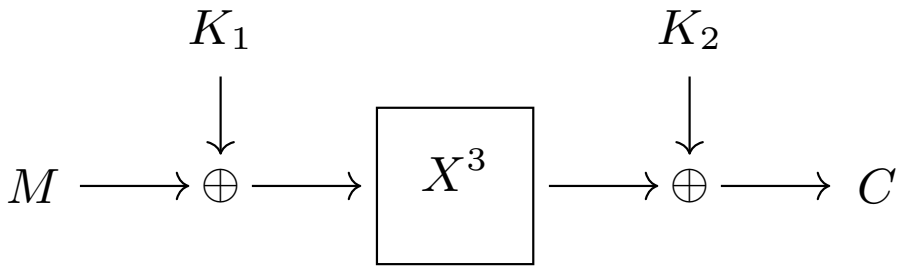
- Example: One round of the MiMC–cipher:



(Number of rounds needs to be large in order to be resistant against interpolation attacks).

Overview, Algebraically Simple Block Ciphers

Several new algebraically simple symmetric ciphers have been designed in recent years. This has in particular been accelerated by the ongoing *STARK-Friendly Hash Challenge* competition[c].

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- Jarvis and Friday (Ashur and Dhoogle, 2018) **broken**
- Starkad and Poseidon (Grassi et al., 2019)
- Vision and Rescue (Aly et al., 2019)
- GMiMC (Albrecht et al., 2019)

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Security of these ciphers are not well understood. Some recent relevant analysis:

- Improved interpolation on round–reduced MiMC (Li–Preneel, SAC'19)
- Cryptanalysis of Jarvis and Friday with Gröbner bases (Albrecht et al., Asiacrypt'19)
- Attack on full–round MiMC (Strong Attack Model). Current joint work with L. Grassi, C. Rechberger, and M. Schofnegger.

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Bilinear Polynomial Systems

- Let $x_1, ..., x_{n_1}$ and $y_1, ..., y_{n_2}$ be two sets of variables. Then a bilinear polynomial in these two sets can be written on the form:

$$p(x_1, ..., y_{n_2}) = \sum a_{i,j} x_i y_j + \sum b_i x_i + \sum c_j y_j + d, \quad \text{for } a, b, c, d \in \mathbb{F}.$$

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- Bilinear systems may for example arise in cryptography through matrix multiplication:

$$0 = XY = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \begin{bmatrix} y_1 & y_2 \\ y_3 & y_4 \end{bmatrix} = \begin{bmatrix} x_1y_1 + x_2y_3 & x_1y_2 + x_2y_4 \\ x_3y_1 + x_4y_3 & x_3y_2 + x_4y_4 \end{bmatrix} = \begin{bmatrix} p_1 & p_2 \\ p_3 & p_4 \end{bmatrix}.$$

Solving Bilinear Systems

- (Faugère et al., 2011) showed that the left kernel of the jacobian matrix of the polynomial system (w.r.t the x– or y–variables) will be in the ideal of the system.

$$Jac_X(\mathcal{P}) = \begin{bmatrix} \frac{\partial p_1}{\partial x_1} & \frac{\partial p_1}{\partial x_2} & \cdots & \frac{\partial p_1}{\partial x_{n_x}} \\ \frac{\partial p_2}{\partial x_1} & \frac{\partial p_2}{\partial x_2} & \cdots & \frac{\partial p_2}{\partial x_{n_x}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial p_m}{\partial x_1} & \frac{\partial p_m}{\partial x_2} & \cdots & \frac{\partial p_m}{\partial x_{n_x}} \end{bmatrix}$$

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- Furthermore, (Verbel, et al., 2019) and (Bardet et al., 2019) showed that this left kernel has additional structure when the polynomial system is generated by matrix multiplication.
- Combining these works implies that an attacker can pre–calculate this left kernel, which will be polynomials of (relatively) low degree in one of the sets of variables. This avoids reducing huge Macaulay matrices (as e.g. in a straightforward Gröbner basis calculation).

Effects on Rank–Metric Code Based Cryptosystems

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- Currently there are two rank–metric code based encryption systems in the NIST competition, ROLLO and RQC.
- (Bardet et al., 2019) showed that these cryptosystems can be broken by solving certain bilinear polynomial systems.
- In particular, their attack breaks 11 out of the 12 suggested parameter sets of ROLLO and RQC.

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Rank–Metric Code Based Encryption Schemes

- Recent breakthrough in analysis, based on Gröbner basis methods.
- Remains to be seen how much these attacks can be improved, and whether there at the end of the day is possible to choose secure and efficient parameters.

Thank you!