Norwegian University of Science and Technology
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MA8404 Numerical solution of time dependent differential equations
Autumn 2019

Solutions to exercise set 3

1 **a)** We first consider the method

$$y_{n+2} + y_{n+1} - 2y_n = \frac{h}{4}\left[f(t_{n+2}, y_{n+2}) + 8f(t_{n+1}, y_{n+1}) + 3f(t_n, y_n)\right].$$

$$C_0 : \sum_j \alpha_j = 1 + 1 - 2 = 0$$
$$C_1 : \sum_j j\alpha_j - \beta_j = 0$$
$$C_2 : \sum_j \frac{j^2}{2}\alpha_j - j\beta_j =$$
$$C_3 : \sum_j \frac{j^3}{6}\alpha_j - \frac{j^2}{2}\beta_j == 0$$
$$C_4 : \sum_j \frac{j^4}{24}\alpha_j - \frac{j^3}{6}\beta_j == \frac{1}{24} \neq 0$$

From this we see that the method has order 3, its error constant is given by $C_4 = \frac{1}{24}$ and that the method is consistent. For zero-stability we consider the roots of the polynomial

$$r^2 + r - 2 = 0 \qquad \Rightarrow \qquad r = 0,\ r = 2.$$

Since these are not bounded by 1 in amplitude, the method is not zero- stable and hence not convergent.

We next consider the method

$$y_{n+3} + \frac{1}{4}y_{n+2} - \frac{1}{2}y_{n+1} - \frac{3}{4}y_n = \frac{h}{8}\left[19f(t_{n+2}, y_{n+2}) + 5f(t_n, y_n)\right].$$

$$
\begin{aligned}
C_0 &: \sum_j \alpha_j = 0 \\
C_1 &: \sum_j j\alpha_j - \beta_j = 0 \\
C_2 &: \sum_j \frac{j^2}{2}\alpha_j - j\beta_j = 0 \\
C_3 &: \sum_j \frac{j^3}{6}\alpha_j - \frac{j^2}{2}\beta_j = 0 \\
C_4 &: \sum_j \frac{j^4}{24}\alpha_j - \frac{j^3}{6}\beta_j = \frac{17}{48} \neq 0
\end{aligned}
$$

From this we see that the method has order 3, its error constant is given by $C_4 = \frac{17}{48}$ and that the method is consistent. For zero-stability we consider the roots of the polynomial

$$r^3 + \frac{1}{4}r^2 - \frac{1}{2}r - \frac{3}{4} = 0 \qquad \Rightarrow \qquad r = 1,\ r = \frac{-5}{8} \pm \frac{\sqrt{23}}{8}i.$$

Since these are bounded by 1 in amplitude, the method is zero- stable. Hence it is a convergent method of order 3.

We finally consider the method

$$y_{n+2} - y_{n+1} = \frac{h}{3}\left[3f(t_{n+1}, y_{n+1}) - 2f(t_n, y_n)\right].$$

$$
\begin{aligned}
C_0 &: \sum_j \alpha_j = 0 \\
C_1 &: \sum_j j\alpha_j - \beta_j = \frac{-2}{3} \neq 0
\end{aligned}
$$

From this we see that the method has order 0, its error constant is given by $C_1 = \frac{-2}{-3}$ and that the method is not consistent. For completeness, we still investigate the zero- stability property of the method. We consider the roots of the polynomial

$$r^2 - r = 0 \qquad \Rightarrow \qquad r = 0,\ r = 1.$$

So the method is zero- stable. But as already metioned it is not convergent since it is not consistent.

2 **a)** After all this calculations, the result should be

$$p'(t_{n+2}) = \frac{1}{h}\left(\frac{1+2\omega}{(1+\omega)\omega}y_{n+2} - \frac{1+\omega}{\omega}y_{n+1} + \frac{\omega}{1+\omega}y_n\right) = f(t_{n+2}, y_{n+2})$$

or

$$y_{n+2} - \frac{(1+\omega)^2}{1+2\omega}y_{n+1} + \frac{\omega^2}{1+2\omega}y_n = h\frac{\omega(1+\omega)}{1+2\omega}f(t_{n+2}, y_{n+2}).$$

b) The local discretization error is

$$\begin{aligned} h\tau_{n+2} &= y(t_n + (1+\omega)h) - \alpha_1(\omega)y(t_n + h) + \alpha_0(\omega) - h\beta(\omega)y'(t_n + (1+\omega)h) \\ &= -\frac{1}{6}\frac{(1+\omega)^2\omega^2}{2\omega+1}h^3 y'''(t_n) + \mathcal{O}(h^4). \end{aligned}$$

The characteristic polynomial is given by

$$\rho(r) = r^2 - \frac{(1+\omega)^2}{1+2\omega}r + \frac{\omega^2}{1+2\omega}$$

with roots

$$r = 1,\ r = \frac{\omega^2}{1+2\omega}$$

meaning that the method is zero-stable, and thus convergent, only for $\omega < 1 + \sqrt{2}$.

c) See the enclosed Jupyter notebook.