Norwegian University of Science and Technology
Department of Mathematical Sciences

MA8404 Numerical solution of time dependent differential equations
Autumn 2019

Solutions to exercise set 1

1 Before starting: The order conditions up to order 3 is given by:

$$p = 1, \qquad \sum_i b_i = 1,$$
$$p = 2, \qquad \sum_i b_i c_i = \frac{1}{2},$$
$$p = 3, \qquad \sum_i b_i c_i^2 = \frac{1}{3}, \qquad \sum_{ij} b_i a_{ij} = \frac{1}{6},$$

where $c_i = \sum_{j=1} a_{ij}$. Thus $a_{21} = c_2 - \gamma$.

a)
- All order 2 methods satisfies

$$b_1 = \frac{c_2 - \frac{1}{2}}{c_2 - \gamma}, \qquad b_2 = \frac{\frac{1}{2} - \gamma}{c_2 - \gamma}.$$

- With these values of b_i the last order 4 condition is reduced to

$$\gamma^2 - \gamma + \frac{1}{6} = 0 \qquad \Rightarrow \gamma = \frac{1}{2} \pm \frac{\sqrt{3}}{6},$$

and the c_2 satisfying the first of the order 3 conditions becomes

$$c_2 = \frac{1}{2} \mp \frac{\sqrt{3}}{6}.$$

Altogether, the two methods becomes:

$$\begin{array}{c|cc} \frac{1}{2} \mp \frac{\sqrt{3}}{6} & \frac{1}{2} \mp \frac{\sqrt{3}}{6} & 0 \\ \frac{1}{2} \pm \frac{\sqrt{3}}{6} & -\frac{\sqrt{3}}{3} & \frac{1}{2} \mp \frac{\sqrt{3}}{6} \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

- For the methods to be stiffly accurate we require $c_2 = 1$, and then $b_1 = 1 - \gamma$, $b_2 = \gamma$, which is satisfied for all γ's satisfying

$$\gamma^2 - 2\gamma + \frac{1}{2} = 0 \qquad \gamma = \frac{1}{2} \mp \frac{\sqrt{3}}{6},$$

resulting in the methods

$$\begin{array}{c|cc} 1 \mp \frac{\sqrt{2}}{2} & 1 \mp \frac{\sqrt{2}}{2} & \\ 1 & \pm\frac{\sqrt{2}}{2} & 1 \mp \frac{\sqrt{2}}{2} \\ \hline 1 & \pm\frac{\sqrt{2}}{2} & 1 \mp \frac{\sqrt{2}}{2} \end{array}$$

b) In general, each stage of an SDIRK method applied to the linear test equation $y' = \lambda y$ can be expressed as

$$Y_i = (1-\gamma z)^{-1}(y_n + \sum_{j=1}^{i-1} a_{ij} Y_j), \qquad i = 1, 2, \ldots, s.$$

thus the stability functions for these methods will always be something like

$$R(z) = \frac{P(z)}{(1-\gamma z)^s},$$

where $P \in \mathbb{P}_s$. For $s = 2$ the general order 2 method has the following stability function:

$$R(z) = \frac{(\gamma^2 - 2\gamma + \frac{1}{2})z^2 - (2\gamma - 1)z + 1}{(1-\gamma z)^2}.$$

and then it is just to plug in the values for γ. Notice that for the stiffly accurate methods, the z^2 term of the numerator is 0, thus $R(z) \to 0$ whenever $|z| \to \infty$.

c) The method is A-stable if

- $|R(iy)| \le 1$ for all real y,
- All poles of $R(z)$ are in $\mathbb{C}^+$.

The latter is obviously true for $\gamma \ge 0$. The first is true if

$$E(y) = |Q(iy)|^2 - |P(iy)|^2 \le 0, \quad \text{for all real } y,$$

when $R(z) = P(z)/Q(z)$. Which in our case simply becomes

$$E(y) = y^4 \left(4\gamma^3 - 5\gamma^2 + 2\gamma - \frac{1}{4}\right).$$

The methods are A-stable for $\gamma \ge 1/2$. In conclusion, the order 3 method is A-stable only for $\gamma = \frac{1}{2} + \frac{\sqrt{3}}{6}$, the stiffly accurate methods are both A-stable (and L-stable). However, when $\gamma = 1 + \frac{\sqrt{2}}{2}$, get $c_1 > 1$ which means that we use function evaluation taken outside the step, which should be avoided.

d) See the enclosed python file.

2 **a)** For the definition of an algebraically stable method, see HW, p.182, def. 12.5 (and the preceeding theorem). Let $B = \text{diag}(b_1, b_2, \ldots, b_s)$ and

$$M = BA + A^T B - bb^T$$

That M is nonnegative definite, means that

$$x^T M x \ge 0 \quad \text{for all } x.$$

Let $x = [1, 0, \ldots, 0]^T$. For a method with an explicit first stage ($a_{1j} = 0$)

$$x^T M x = -b_1^2 < 0$$

if $b_1 \ne 0$. So no, the method is not B-stable.

b) Of the order three methods: If $\gamma = \frac{1}{2} + \frac{\sqrt{3}}{6}$, the method is B-stable, if $\gamma = \frac{1}{2} - \frac{\sqrt{3}}{6}$, it is not (check the sign of the eigenvalues of the M-matrix).
None of the stiffly accurate order 2 methods are B-stable.

3 See HW, p. 181, Example 12.3.

4 Given

$$\begin{aligned} y_1' &= y_1 & y_1(0) &= 1 \\ y_2' &= z - y_2, & y_2(0) &= 0 \\ z' &= z + y_2 - 2w & z(0) &= 1 \\ 0 &= y_1 - e^{y_2}, & w(0) &= 0 \end{aligned}$$

Differentiate the algebraic constraint:

$$\begin{aligned} 0 &= y_1' - e^{y_2} y_2' = y_1 - exp(y_2)(z - y_2), \\ 0 &= y_1' - e^{y_2} y_2'(z - y_2) - e^{y_2}(z' - y_2') = y_1 - e^{y_2}((z - y_2)^2 + 2y_2 - 2w), \end{aligned}$$

and the latter can obviously be solved wrt w, and is, together with the first three differential equations an index 1 problem. So we have differentiated the algebraic constraint twice to get an index 1 DAE, the index of the original problem is then 3.

The index could also be found by writing the equation in Hessenberg form, with

$$y' = f(y, z), \qquad z' = g(y, z, w), \qquad h(y) = 0$$

and using that

$$h_y f_z g_w = \begin{bmatrix} 1 & -e^{y_2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} -2 \end{bmatrix} = 2e^{y_2} \neq 0.$$

The solution is, starting with the first equation:

$$y_1 = e^t, \quad y_2 = t, \quad z = t + 1, \qquad w = t.$$

5 **a)** Differentiate the first equation, and subtract the two:

$$z(t) = g(t) - f'(t), \qquad y(t) = f(t) - \eta t(g(t) - f'(t)).$$

b) Backward Euler becomes

$$\begin{aligned} y_{n+1} + \eta t_{n+1} z_{n+1} &= f(t_{n+1}) \\ \frac{y_{n+1} - y_n}{h} + \eta t_{n+1} \frac{z_{n+1} - z_n}{h} + (1 + \eta) z_{n+1} &= g(t_{n+1}). \end{aligned}$$

Using that $y_n + \eta t_n z_n = f(t_n)$ from the previous step, and $t_{n+1} = t_n + h$, we can show that

$$z_{n+1} = \frac{\eta}{1 + \eta} z_n + g(t_{n+1}) - \frac{f(t_{n+1} - f(t_n)}{h}.$$

The last two terms are as expected. But the first term will only go to 0 for $n \to \infty$ if $\eta > -1/2$, if $\eta < -1/2$ the solution will diverge.

c) The transformed problem will simply be

$$\hat{y} = f(t), \qquad \hat{y}' + \hat{z} = g(t)$$

for which Backward Eulers method becomes

$$\hat{y}_{n+1} = f(t_{n+1}), \qquad \hat{z}_{n+1} = g(t_{n+1}) - \frac{f(t_{n+1}) - f(t_n)}{h}$$

which converges to the exact solution when $h \to 0$.

6 **a)** See the rather poor, but enclosed matlab file.

b) From the matlab file, we got something like:

	DAE	SSF	ODE
Error in ax^2:	$2.8 \cdot 10^{-7}$	$5.2 \cdot 10^{-7}$	$6.2 \cdot 10^{-7}$
Number of steps:	557	197	5467
Number of rejected steps:	84	17	160
Number of function evaluations:	1096	432	10884
Number of jacobians:	12	1	38

So, the state space form (SSF) is trivial to solve, the DAE is reasonable, while the index reduced ODE is about 10 times as hard to solve as the DAE.

c) This is strictly speaking only interesting in the case of the ODE. In that case, the Jacobian is given by

$$J = \begin{bmatrix} -2ax & 1 \\ 2a(y - ax^2 + \cos(t)) - 4a^2x^2 & 2ax \end{bmatrix} \begin{bmatrix} -2ax & 1 \\ -4a^2x^2 + 2a\cos(t) & 2ax \end{bmatrix}$$

for $y = ax^2$, which neatly enough has the eigenvalues $\pm\sqrt{2a\cos(t)}$. This means that there are solutions close to the stable manifold $y = ax^2$ will not be attracted to it, on the contrary. And since the numerical solution will move a bit away from the stable solution, the integrator has to deal with these solutions as well.

In fact, for this simple problem, it is possible to find the exact solutions, which are in some arbitrary point t_0, x_0, y_0:

$$\begin{aligned} x(t) &= (t - t_0)(y_0 - ax_0^2) + x_0 + \sin(t) - \sin(t_0) \\ y(t) &= y_0 - ax_0^2 - ax(t)^2 \end{aligned}$$

which is unstable whenever $y_0 \neq ax_0^2$.