Norwegian University of Science and Technology
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# MA8404 Numerical solution of time dependent differential equations
Autumn 2019

**Exercise set 5**

**1** Given the stochastic differential equation:

$$\mathrm{d}X = -\alpha X \mathrm{d}t + \sigma \mathrm{d}W(t), \qquad X(0) = x_0 \tag{1}$$

where $\alpha, \sigma \in \mathbb{R}$ are some constants, $\alpha > 0$. For simplicity, we also assume that the initial value $x_0$ is a constant (and not a stochastic variable).

**a)** Prove that the exact solution of this SDE is given by

$$X(t) = \mathrm{e}^{-\alpha t} x_0 + \beta \int_0^t \mathrm{e}^{-\alpha(t-s)} \mathrm{d}W(s)$$

Also, show that this is a Gaussian process with

$$\mathbb{E}X(t) = \mathrm{e}^{-\alpha t} x_0, \qquad \mathrm{Var}X(t) = \frac{\sigma^2}{2\alpha}(1 - \mathrm{e}^{-2\alpha t}).$$

**b)** When (1) is solved numerically by an Euler-Maruyama method, the strong mean square order of the numerical solution is 1, and not 1/2 as expected. Explain why.

Given a Wiener process simulated by

$$W(t_n + h) = W(t_n) + \xi_n \sqrt{h}, \qquad \xi_n \sim \mathcal{N}(0, 1).$$

Then the exact solution of (1) can be simulated by

$$X(t_n + h) = \mu X(t_n) + \xi_n \kappa.$$

where

$$\mu = \mathrm{e}^{-\alpha h}, \qquad \kappa^2 = \frac{\sigma^2}{2\alpha}(1 - \mathrm{e}^{-2\alpha h}).$$

**c)** Solve (1) by the Euler-Maruyama method, and compare the exact and the numerical solution. Verify the order result from point **b)** numerically.

Use e.g. $\alpha = 1$ and $\sigma = 0.5$ in your experiments.

**2** Given an Itô SDE

$$\mathrm{d}X = f(X)\mathrm{d}t + g(X)\mathrm{d}W(t), \qquad X(0) = x_0$$

and the SRK method

$$H_2 = Y_n + \sqrt{h} g(Y_n)$$
$$Y_{n+1} = Y_n + hf(Y_n) + I_1 g(Y_n) + \frac{I_{11}}{\sqrt{h}}(g(H_2) - g(Y_n))$$

where

$$I_1 = \int_{t_n}^{t_{n+h}} \mathrm{d}W(s) = \Delta W_n, \qquad I_{11} = \int_{t_n}^{t_n+h} \int_{t_n}^{t_n+s} \mathrm{d}W(s_1)\mathrm{d}W(s) = \frac{1}{2}(\Delta W_n^2 - h).$$

**a)** Write up the stochastic Butcher tablaux for the method, and find its strong order.

**b)** Discuss the MS-stability properties of the method, applied to the linear test equation

$$dX = \lambda X \mathrm{d}t + \mu X \mathrm{d}W(t)$$

with $\lambda, \mu$ being real constants.

**3** (optional) The following is example is taken from L.Cobb, *Stochastic differential equations for the Social Sciences*, in Mathematical Frontiers of the Social and Policy Sciences, Westview Press, 1981. It seems to still have some relevance.

"*Let $x_t$ be a person's political persuasion on the liberal-conservative dimension, where $x = 0$ is an extremely 'liberal' conviction and $x = 1$ is an extremely 'conservative' conviction. Suppose further that there is a general tendency to move toward the average persuasion of the whole population, but that people who hold extreme views are much less subject to random fluctuations than are those near the center. Thus $\sigma^2(x) = \varepsilon x(1-x)$, and*

$$dx_t = r(G - x_t)dt + \sqrt{\varepsilon x_t(1 - x_t)}dw_t$$

*describes the motion of each person through the political spectrum. Note that in statistical equilibrium the average of $x$ will be $G$, no matter what values $r$ and $\varepsilon$ have. The coefficient $r$ in equation (11) expresses the strength of the general tendency to conform: we may reasonably assume that this is a constant. On the other hand, the coefficient $\varepsilon$ measures the amount of random change occurring in political persuasion, and this is certainly large during times of unrest and small during times of political tranquility. What happens to the shape of the distribution of political persuasion as $\varepsilon$ changes?* "

Yes, what indeed? Solve the equation, plot the distribution after some time, and give some interpretation of the result.
Cobb uses $r = 1$ and $G = 0.7$, and 4 unspecified values of $\varepsilon$.

(I have not tried it out myself, so your choices of free parameters are as good as mine.)