Norwegian University of Science and Technology
Department of Mathematical Sciences

MA8404 Numerical solution of time dependent differential equations
Autumn 2019

Exercise set 1

1 *About the Lipschitz condition, the one-sided Lipschitz-condition, and logarithmic norms.*

In this exercise, only the 2-norm is considered. We will also for simplicity assume that all conditions are valid for all $y, v \in \mathbb{R}^m$ and all relevant intervals $t \in (t_0, t_{end})$.

a) Consider the system of equations

$$\begin{aligned} y_1' &= -100y_1 + y_2, \qquad & y_1(0) &= 1 \\ y_2' &= y_1 - 100y_2, \qquad & y_2(0) &= 0 \end{aligned} \tag{1}$$

Find the exact solution of the problem.

For ODEs $y' = f(t, y)$ we have the following well known result:

$$\|f(t,y)-f(t,v)\| \leq L\|y-v\| \quad \Rightarrow \quad \|y(t)-v(t)\| \leq e^{L(t-t_0)}\|y(t_0)-v(t_0)\|, \qquad t > t_0$$

b) Use this to find a bound for $\|y(1)-v(1)\|_2$ when $v(t)$ is the solution of the ODE (1) with initial values $v_1(0) = 1, v_2(0) = 1/10$.

The function $f(t, y)$ satisfies a *one-sided Lipschitz condition* if, for all t,

$$\langle f(t,y) - f(t,v), y - v\rangle_2 \leq \mu\|u - v\|_2^2, \qquad \forall u, v \in \mathbb{R}^m. \tag{2}$$

where μ is called a one-sided Lipschitz constant. It can be proved that if (2) is satisfied, then

$$\|y(t) - v(t)\|_2 \leq e^{\mu(t-t_0)}\|y(t_0) - v(t_0)\|_2, \qquad t > t_0 \tag{3}$$

see Theorem 2.4 in the *Lecture notes on the numerical solution of ODEs*. Further, it can be proved that if $f(t, y) = Ay$, then the smallest possible one-sided Lipschitz constant for f is the logarithmic norm

$$\mu_2(A) = \lim_{h\to 0, h>0} \frac{\|I + hA\|_2 - 1}{h} \tag{4}$$

and in particular that

$$\mu_2(A) = \lambda_{\max}\left(\frac{A + A^T}{2}\right) \tag{5}$$

c) Use (3) to find a better bound for $\|y(1)-v(1)\|_2$ of **b)**. Compare with the exact solution.

d) (Optional) Prove the statements related to (4) and (5).

2 *About the practical use of B-series and rooted trees*

a) Kutta's method from 1901 is definitely the most famous of all explicit Runge–Kutta pairs. Its Butcher tableau is given by

$$\begin{array}{c|cccc} 0 & & & & \\ \frac{1}{2} & \frac{1}{2} & & & \\ \frac{1}{2} & 0 & \frac{1}{2} & & \\ 1 & 0 & 0 & 1 & \\ \hline & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} \end{array}$$

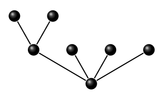
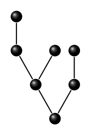
Verify that the method is of order 4 by confirming that it obeys all the eight order conditions.

b) Show that an explicit Runge–Kutta method of order 3 with 3 stages has to satisfy

$$3\,a_{32}\,c_2^2 - 2\,a_{32}\,c_2 - c_2\,c_3 + c_3^2 = 0. \tag{6}$$

c) Characterize all 3rd-order explicit Runge–Kutta methods with 3 stages which satisfy $a_{31} = 0$, that is, which satisfy $a_{32} = c_3$. How many free parameters are there?

d) Write down the order conditions corresponding to the following rooted trees:

Also, write down the order of the trees.
Hint: Use the result (7) of the next exercise.

3 In the lecture note, it is proved that the B-series of the exact solution of the ODE $y' = f(y)$ can be written as

$$y(t_0 + h) = B(e, y_0; h), \qquad \text{with} \qquad e(\tau)(h) = \frac{1}{\gamma(\tau)} h^{\rho(\tau)}.$$

Show that

$$\gamma(\bullet) = 1, \qquad \gamma(\tau) = \rho(\tau) \prod_{i=1}^{\kappa} \gamma(\tau_i), \qquad \text{when} \quad \tau = [\tau_1, \tau_2, \ldots, \tau_\kappa]. \tag{7}$$

At this point, it could also be a good idea to compare our definition of B-series with the definition used in GNI, III.1, def.1.8.

4 *For a better understanding of the B–series*

a) Consider ODEs of the form

$$y' = y^2$$

For which trees $\tau \in T$ will the elementary differentials $F(\tau)(y) = 0$?
What if

$$y' = Ay + g(t), \qquad A \text{ constant matrix?}$$

b) (Optional) Derive a B-series for the exact solution of a split ODE:

$$y' = f(y) + g(y).$$

Hint: You may need bi-colored trees, one color for each of f and g.