MA8403 Week 4

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1. Question 1

Show that if $(\mathcal{L}, \mathcal{R})$ is a weak factorization system on $\mathcal{C}$, then both $\mathcal{L}$ and $\mathcal{R}$ are closed under forming retracts.[1] Deduce that in a model category, the class of fibrations and cofibrations are closed under retracts.

2. Question 2

In this exercise, we will show that in a model category, the composite of cofibrations is a cofibration. The dual argument shows the same for fibrations. Let $\mathcal{C}$ be a model category.

1. Show that if $f \colon X \to Y$ can be factored as $f = pi$ where p has the right lifting property with respect to f, then f is a retract of i. Dually, if f can be factored as $f = pi$ where i has the left lifting property with respect to g, then g is a retract of p.
2. Deduce that a map $i \colon A \to B$ is a cofibration if and only if it has the left lifting property with respect to all trivial fibrations.
3. Conclude that if $\mathcal{C}$ is a model category, then the class of cofibrations is closed under compositions.

3. Question 3

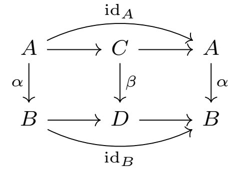
Recall that the Borel construction $EG \times_G X$ on a (left) G-space X is defined as the orbits with respect to the diagonal G-action on $EG \times X$.

1. If Y is a left G-space, show that the prescription $y \cdot g \coloneqq g^{-1} \cdot y$ defines a right G-action on Y.
2. For a right G-space Y and a left G-space X, define an equivalence relation on $Y \times X$ by

$$(y \cdot g, x) \sim (y, g \cdot x).$$

   Show that the quotient space $(Y \times X)/\sim$ is homeomorphic to $Y \times_G X$, where we are using (1) to consider Y as either a left or right G-space.

[1]If there is a commutative diagram



then we will say that the map α is a retract of the map β. In other words, there is a retract in the arrow category $\mathcal{C}^{\Delta[1]}$.