Lie groups and Lie algebras, Spring 2013, NTNU

Here are some basic concepts and topics which the student is encouraged to study and explain. By a field $\mathbf{k}$ we shall mean $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ (quaternions). Note that $\mathbb{H}$ is not commutative.

- What is a group? If we say that (G, M) is a transformation group of the set M, what does this mean ?
- If V is a vector space (over some field, say $\mathbb{R}$ or $\mathbb{C}$), what group is $GL(V)$? What group is usually denoted $GL(n,\mathbb{R})$, or sometimes $GL_{\mathbb{R}}(n)$, $GL_n(\mathbb{R})$?
- What is a linear group (over a given field) ? What is a matrix group? Why are these concepts essentially the same ?
- Describe the following classical matrix groups (over some $\mathbf{k}$) :

$$\begin{aligned} SO(n), n &= 2, 3, ... \\ O(n), n &= 1, 2, 3, ... \\ SU(n), n &= 2, 3, ... \\ U(n), n &= 1, 2, 3, ... \\ Sp(n), n &= 1, 2, 3, ... \\ SL(n, \mathbb{R}), n &= 2, 3, ... \end{aligned}$$

  Describe the properties of their matrices, and also in terms of bilinear form.

- What is the general definition of a Lie group G. (You need to know what a smooth manifold is).
- What is a homomorphism $\varphi : G \to H$ between two Lie groups.
- Explain what is a (linear!) representation of a Lie group G on a vector space V.
- What is a Lie algebra ? (explain the algebraic structure). Explain why 3-space $(\mathbb{R}^3, \times)$ with the usual cross product $\mathbf{u} \times \mathbf{v}$ as Lie bracket $[u, v]$ is a Lie algebra.
- Show that the above Lie algebra $(\mathbb{R}^3, \times)$ is isomorphic to the Lie algebra $so(3)$, consisting of skew-symmetric real 3×3-matrices.
- How would you define the Lie algebra $L(G)$ of a Lie group G?
- If $G = GL(V)$, or $GL(n, \mathbf{k})$, what is the Lie algebra of G?
- What is a 1-parameter group of transformations on a manifold M?

- What is a 1-parameter group of the Lie group G?
- ‘How would you define the exponential map $exp : L(G) \to G$ of a Lie group G? What are the basic properties of this map?
- For the above family of classical (matrix) groups, determine their (matrix) Lie algebras. Use this to determine the dimensions of these groups.
- Decribe the Lie algebra and exponential map of the Lie group $(\mathbb{R}^n, +)$ and the n-dimensional torus $T^n = S^1 \times S^1 \times .... \times S^1$
- Consider a family of vector fields $X_1, ...X_r$ on $\mathbb{R}^n$ (or some open set of $\mathbb{R}^n$). What does it mean (following the ideas of Sophus Lie himself) that the vector fields generate an r-dimensional Lie algebra (Infinitesimal group, in Lies's terminology).
- Howe would you describe a (possibly local) 3-parameter continuous group in the xy-plane, in Lie's terminology?
- Give examples of non-compact and compact connected Lie groups.
- What does Cartan's maximal torus tell us about compact connected Lie groups?
- The notions "irreducible" and "reducible" representations, explain.
- In the case of $SO(3)$, it is well known that the three lowest dimensional real irreducible representations are φ_0, φ_1 and φ_2,of dimension 1, 3,5 resp. Can you describe φ_0, φ_1? In the case of φ_2, as a hint, first observe that there is a well known action (representation) of $SO(3)$ on the space of 3-dim symmetric real matrices.