k alg. closed

8 March

IV : Dimension

[X = top space]

① Topological & algebraic.

Defn: Let X = top space. "length n"

$$\dim X = \sup\{ n \in \mathbb{Z} \mid X_0 \subsetneq \cdots \subsetneq X_n \text{ is a chain of irreducible, closed subsets of } X \}$$

Note: Th.3 is only useful for topologies such as Zariski.
(If X is separated, then $\dim X = 0$.)

Prop 1.3: Let $Y \subseteq X$ be a subspace of X.

① $\dim Y \leq \dim X$

② If X irred, $\dim X < \infty$, $Y \overset{\text{closed}}{\subsetneq} X$, then $\dim Y < \dim X$.

Pf:

① Let $F_0 \subsetneq \cdots \subsetneq F_n$ be a chain of closed, irred subsets of Y.

Then: $\overline{F}_0 \subsetneq \cdots \subsetneq \overline{F}_n$ is a chain of closed subsets of X.

distinct: $\overline{F}_i \cap Y = F_i$ ($\overline{F}_i = \overline{F}_{i+1} \Rightarrow F_i = F_{i+1}$ $\lightning$)

irreducibility:

If $V \subseteq Y \subseteq X$ is irred closed in Y, then $\overline{V}$ irred.

Suppose $\overline{V} = F \cup G$ (ie $\overline{V}$ not irred)

(F, G closed, nonempty.)

Then $V = \overline{V} \cap Y = (F \cap Y) \cup (G \cap Y)$

($F \cap Y$ closed, $G \cap Y$ closed)

WLOG, $G \cap Y = \emptyset$. $\Rightarrow V \leq F$

Since $\overline{V} = \bigcap$ closed sets containing V $\Rightarrow G = \emptyset$ $\lightning$. //

$\therefore \dim Y \leq \dim X$.

② Assume X irred, $\dim X < \infty$, $Y \subsetneq X$ closed

Sng $\dim Y = n$.

Then a max length chain in Y looks like

$$F_0 \subsetneq \cdots \subsetneq F_n \subseteq Y.$$

Thus $\overline{F_0} \subsetneq \cdots \subsetneq \overline{F_n} \subsetneq X \Rightarrow \dim X > n.$ //

(chain in X.)

TRY: Find (lower bounds) for dimensions:

k^n, $\operatorname{Spec}\mathbb{Z}$, $\operatorname{Spec} k[X_1, \ldots, X_n]$.

k^2: $(0) \subsetneq V(Y) \subsetneq k^2$ (the x-axis)

$\operatorname{Spec}\mathbb{Z}$: $(0) \subsetneq (p)$, $\dim \operatorname{Spec}\mathbb{Z} \geq 1$

$\operatorname{Spec} k[X_1, \ldots, X_n]$: $(0) \subsetneq (X_1) \subsetneq (X_1, X_2) \subsetneq \cdots \subsetneq (X_1, \ldots, X_n)$ (length n of prime ideals)

$\dim k^n \geq n$.

$V(0) \supsetneq V((X_1)) \supsetneq \cdots \supsetneq V(X_1, \ldots, X_n)$

$\operatorname{Spec}\mathbb{Z} = \{\text{prime ideals}\}$

$V(p) = \{q \in \operatorname{Spec}\mathbb{Z} \mid q \supseteq p\}$ ← AM.

↑ closed sets in Zariski top

Prop 1.4: If $X = \bigcup_{i=1}^{n} X_i$, X_i closed in X.

then $\dim X = \sup \dim X_i$.

Pf: $\boxed{\geq}$ follows since $\dim X \geq \dim X_i$ $\forall i$.

$\leq$ Let $p = \sup\{\dim X_i\} < \infty$
↑ may assume (else trivial).

Suppose $\dim X > p$.

Then we may find a chain

$$F_0 \subsetneq \cdots \subsetneq F_p \subsetneq F_{p+1}$$ ↑ closed, irred subset of X.

Thus $F_{p+1} = \bigcup_{i=1}^{n} (F_{p+1} \cap X_i)$ ⇒ $F_{p+1} = F_{p+1} \cap X_i \subseteq X_i$ some i.
(closed)

⇒ X_i has chain length $p+1$, ie. $\dim X_i \geq p+1$ $\lightning$.

Recall: If X = algebraic variety, then $X = \bigcup_{i=1}^{n} F_i$, F_i irred, closed, not containing each other.
↑ these are the irreducible components.

Thus: we usually reduce to looking dim of irred. alg. varieties.

Relation with Krull dimension.

For any ring A, the Krull dimension is

$$\text{Krull.dim } A = \sup\{n \in \mathbb{Z} \mid p_0 \subsetneq \cdots \subsetneq p_n \text{ prime ideals}\}.$$
length

Prop: Let V be an affine algebraic variety. Then $\dim V = \text{Krull.dim } \Gamma(V)$ $\left(\Gamma(V) = \mathcal{O}_V(V) = k[X_1,\dots,X_n]/I(V)\right)$

Pf: There is a correspondence: (using Nullstellensatz).

$$\begin{array}{ccc} V_n & \longleftrightarrow & I(V_n) \quad \text{prime.} \\ \cup\!| & & \cap\!| \\ \vdots & \vdots & \vdots \\ \cup\!| & & \cap\!| \\ V_0 & \longleftrightarrow & I(V_0) \end{array}$$

V_0: closed, irred in V

Ex:

$V = k^n$ $\qquad \Gamma(V) = k[X_1,\dots,X_n]$

closed irred subsets. $\left\{\right.$ $V(X_1) \longleftrightarrow (X_1)$ $\left.\right\}$ primes ideals.

$\cup\!|$ $\qquad\qquad \cap\!|$

$V(X_1,X_2) \longleftarrow (X_1,X_2)$

$\vdots \qquad\qquad \vdots$

$\dim k^n \geq n \qquad$ Krull. $\dim \geq n$

$\uparrow \qquad\qquad \uparrow$

$=$

GOAL: Show $\dim k^n \left(= \text{Krull_dim } k[X_1,\dots,X_n]\right) = n$.

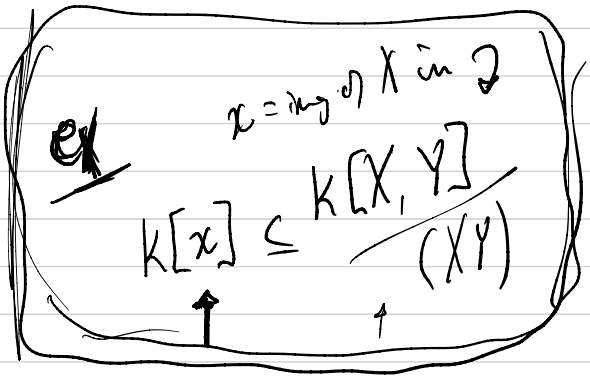
From comm alg:

$\left(k[X_1,\dots,X_n]/I\right)$

Noether's normalization lemma

Let A be a f.g. k-algebra.
There exists $y_1,\dots,y_r \in A$ which are algebraically independent over k and such that A is integral over $k[y_1,\dots,y_r]$.

integral: all elements of A satisfy a monic polynomial with coeffs in $k[y_1,\dots,y_r]$.



[see ch. 5 AM for review]

(Note: this r is going to be dim)

Thm 1.8 (Dimension Theorem)

Let A be a domain that is a finite type k-algebra.
Then $\text{Krull.dim } A = \text{tr.deg}_k \text{Fr}(A)$

$A_{(0)}$ = $\text{Fr}(A)$

transcendance degree of $\text{Fr}(A)$ over k.

Pf: Noether normalization:

$$k[y_1,\dots,y_r] \xhookrightarrow[\text{f.g.}]{} A$$

$$k \longrightarrow k(y_1,\dots,y_r)$$

tr.deg = r.

Since A f.g. domain over $k[y_1,\dots,y_r]$,
then "Going-up theorem" $\Rightarrow$ Krull dim A = Krull.dim $k[y_1,\dots,y_r]$

Enough to consider $k[y_1, \dots, y_r]$.

GOAL: Show $k[y_1, \dots, y_r]$ has no chain of prime ideals of length $> r$.

Induct on r

- $r = 0$ OK.
- $r > 0$

Suppose $\exists$ chain $p_0 \subsetneq \cdots \subsetneq p_m$ with $m > r$.

Choose an elmt $a_1 \in p_1 - p_0$

Since $a_1 \in k[y_1, \dots, y_r]$ is not constant polynomial, a Lemma implies $\exists$ $a_2, \dots, a_r \in k[y_1, \dots, y_r]$ such that $k[a_1, \dots, a_r] \subseteq k[y_1, \dots, y_r]$ f.g.

(*) Nagata/Noether. See notes of Vakil.

$z_i \mapsto a_i$

$$k[z_1, \dots, z_r] \xrightarrow[\varphi]{\text{f.g.}} k[y_1, \dots, y_r]$$

$$k[z_2, \dots, z_r] \cong \frac{k[z_1, \dots, z_r]}{(z_1)} \xrightarrow[\bar{\varphi}]{\text{f.g.}} \frac{k[y_1, \dots, y_r]}{p_1}$$

($r-1$ vars)

$q_1 \subsetneq \cdots \subsetneq q_m \longmapsto \bar{p}_1 \subsetneq \bar{p}_2 \subsetneq \cdots \subsetneq \bar{p}_m$

"Going-down theorem"

$q_1 \subsetneq \cdots \subsetneq q_m$: length $m-1$.

Induction $\Rightarrow$ $m - 1 \leq r - 1$ $\Rightarrow$ $m \leq r$ $\lightning$.

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Cor: In particular, Krull. dim $k[X_1, \dots, X_n] = n$.

Cor: V irred affine alg. variety $\Rightarrow$ $\dim V < \infty$.

Cor: $\dim k^n \left(= \text{Krull-dim } S/k^n\right) = n$.