18 Jan. $\boxed{k = \text{field}}$

I. Affine algebraic sets

① $n \geq 1$ int

Defn: Let $S \subseteq k[X_1, \ldots, X_n]$ be subset.

Set $V(S) = \{(x_1, \ldots, x_n) \in k^n \mid \forall P \in S,\ P(\underline{x}) = 0\}$

$V(S)$ is affine alg. variety of S

Examples

① $V(\{1\}) = \emptyset$

② $V(\{0\}) = k^n$.

③ $n = 1$, $S = \{P\}$.
$V(S)$ finite

④ $n = 2$, the affine subsets of k^2

are
- plane
- $\emptyset$
- curves $V(P)$
- finite sets of points.

Notes ① $S \subseteq S' \Rightarrow V(S) \supseteq V(S')$

② $V(S) = V((S))$

$\supseteq$ clear.

$\subseteq$! $x \in V(S) \Rightarrow \forall f \in S, f(x) = 0$

$(S) \subseteq k[X_1, \ldots, X_n] = R$

For $\sum \underset{\in R}{a_i} \underset{\in S}{f_i} \in (S)$ ....

Prop: Every affine alg. set is defined by finite # of eqns.

$$V(S) = V(f_1, \dots, f_r) \quad \Big(= \underbrace{V(f_1) \cap \dots \cap V(f_r)}_{[\text{hypersurfaces}]} \Big)$$

Pf: $k[X_1, \dots, X_n]$ noeth $\Rightarrow (S)$ f.g.

Prop: The affine alg. sets are the closed sets of a topology on k^n. (Zariski top.)

Pf: • $\emptyset$, k^n affine.

• $\bigcap_i V(S_i) = V(\bigcup S_i)$

$\left(\bigcap V(I_i) = V(\sum I_i) \right)$

• $\bigcup_{i=1}^{n} V(S_i)$ is also affine.

$\hookrightarrow$ ETS $V(I) \cup V(J) = V(IJ)$

$\subseteq$: $IJ \subseteq I, J$

$\Rightarrow V(IJ) \supseteq V(I), V(J)$ //

$\supseteq$: Let $x \in V(IJ)$, $x \notin V(I)$.

$\Rightarrow \exists P \in I$ with $P(x) \neq 0$.

For all $Q \in J$, $PQ \in IJ$

$\Rightarrow (PQ)(x) = 0$

$\Rightarrow Q(x) = 0$ ($k[X_1, \dots, X_n]$ domain).

$\Rightarrow x \in V(J)$. //

$R = k[X, Y]$

(X, Y), $(X, Y+1)$, ...

$\cup\!\!\mid \quad \cup\!\!\mid$

(X) (Y) ....

$\cup\!\!\mid \quad \cup\!\!\mid$ Zariski

$(0))$

$\hookrightarrow$ in Spec R.

$(X-a, Y-b) \longleftrightarrow (a,b) \in k^2$

$(X-a) \longleftrightarrow$ line

Note: this is a very diff top on k^n than the usual one.

here: closed sets in k^3: curves, planes, points.

Defn: For $f \in k[X_1, \dots, X_n]$,

$D(f) = k^n \setminus V(f)$ = standard open sets. (basis.)

Ex: int of any two open sets is nonempty.

$$\begin{aligned} D(f) \cap D(g) &= (k^n \setminus V(f)) \cap (k^n \cap V(g)) \\ &= k^n \setminus (V(f) \cup V(g)) \\ &= k^n \setminus V(fg) \neq \emptyset. \end{aligned}$$

since $V(fg) \neq k^n$.

② Ideal of an affine variety

"dual of $V(-)$"

Defn: Let $V \subseteq k^n$ be a subset

The ideal of V is $I(V) = \{ f \in k[x_1, \ldots, x_n] \mid f(x)=0 \ \forall x \in V \}$

$I(V)$ = kernel of the map

$$r : k[X_1, \ldots, X_n] \longrightarrow \mathcal{F}(V, k)$$
$$p \longmapsto p|_V$$

ring of functions $V \to k$

this is a ring hom $\Rightarrow I(V)$ is an ideal.

Set $\Gamma(V)$ = image of r = poly functn on V.

$$\Gamma(V) \cong k[X_1, \ldots, X_n] / I(V)$$

$\hookrightarrow$ affine algebra of V (k-alg of finite type).

Note: $V \subseteq V' \Rightarrow I(V) \supseteq I(V')$

Exercise: ① V vs $V(I(V))$?

② I vs. $I(V(I))$?

$(\subseteq, \supseteq, = ?)$

or

$\mathbb{V}(-)$

$\mathbb{I}(-)$

$I = (Y - X^2) \subseteq k[X, Y]$

$I = (X^2) \subseteq k[X, Y]$

$V = V(Y - X^2) \subseteq k^2$

① $V \subseteq V(I(V))$

$\uparrow$

$=$ if V is affine.

② $I \subseteq I(V(I))$

$\cap\!\!\!\shortmid \quad \sqrt{I} \quad \cup\!\!\!\shortmid$

ex: $(X^2) = I \subsetneq I(V(I)) = (X)$

details = exercise

Ex: $\mathbb{R}$ $\quad I = (X^2+Y^2+1)$

$V(I) = \emptyset$

$I \subsetneq I(V(I)) = k[X_1, \dots, X_n]$

<u>Prop</u>: 1) $I(\emptyset) = k[X_1, \dots, X_n]$

2) If k is <u>infinite</u>, then $I(k^n) = (0)$.

<u>Pf</u>:

Induct on n:

<u>$n=1$</u>:

$I(k) = \{ f \in k[X_1] \mid f(x) = 0 \ \forall x \in k \}$

finite = false!

$k = \mathbb{Z}/3\mathbb{Z}$

$X^3 - X = 0$

nonzero poly in X_1 has finite # of roots

$\Rightarrow k$ infinite $\Rightarrow [f \in I(k) \Rightarrow f \equiv 0.]$

$n > 1$: Claim $I(k^n) = (0)$. ~~$0 \neq f \in I(k^n)$~~

~~Suppose not, i.e.,~~ Suppose $f \in k[X_1, \ldots, X_n] \setminus \{0\}$ nonconstant.

Write $f = a_r(X_1, \ldots, X_{n-1}) X_n^r + \ldots$ $\quad a_r \neq 0$, $r \geq 1$

Induction $\Rightarrow \exists\, x_1, \ldots, x_{n-1}$ s.t. $a_r(x_1, \ldots, x_{n-1}) \neq 0$

$\Rightarrow f(x_1, \ldots, x_{n-1}, X_n)$ has at most r roots.

$\Rightarrow$ not zero for every $x_n \in k$ (k infinite).

$\Rightarrow f \notin I(k^n)$. //

Exercise: $I(\{(a_1, \ldots, a_n)\}) = (X_1 - a_1, \ldots, X_n - a_n)$.

$\supseteq$ easier.

Open-ended exercise

Compute $I(-)$ and $V(-)$ of various sets/ideals.

k^n $\quad$ or $\quad$ $k[X_1, \ldots, X_n]$.

ch. I ex #2