EXERCISES FOR MA8203 ALGEBRAIC GEOMETRY

UPDATED LAST: APRIL 29, 2021

The numbered exercises come from Daniel Perrin's textbook.
(★) indicates exercise that may be asked on final exam (added 21/4)

Chapter I Exercises. Due 1 February

- (★) 1
- (★) 2
- (★) 7, assume k is infinite
- 8

Chapter II Exercises. Due 1 March

- (★) 1
- (★) 5: you can use Macaulay2 to help find the resolution in part (c)

Chapter III Exercises. Due 15 March

- Exercises about sheaves from class.
  (1) (★) Let $\mathcal{F}$ be a sheaf on a topological space X and let $U \subseteq X$ be open. Show that $\mathcal{F}|_U$ is a sheaf on U.
  (2) (★) Let $\pi : X \to Y$ be a continuous map of topological spaces and let $\mathcal{F}$ be a sheaf on X. Show that $\pi_*\mathcal{F}$, defined by $\pi_*\mathcal{F}(V) = \mathcal{F}(\pi^{-1}(V))$, is a sheaf on Y.
  (3) (★) Let $\mathcal{F}$ be a sheaf of rings on a topological space X.
    (a) If $p \in X$, show that $\mathcal{F}_p$ is a ring.
    (b) If $(X, \mathcal{O}_X)$ is a ringed space, and $\mathcal{F}$ is an $\mathcal{O}_X$-module, show that $\mathcal{F}_p$ is an $\mathcal{O}_{X,p}$-module.
  (4) (★) Show that a sequence $0 \to \mathcal{F} \to \mathcal{G}$ of sheaves of abelian groups on X is exact if and only if $0 \to \mathcal{F}(U) \to \mathcal{G}(U)$ is exact for every open set $U \subseteq X$. Bonus: give an example to show a similar statement is not true for surjections. (corrected 14/3)
- Exercises about algebraic varieties from class. (added 3/3)
  (5) (★) Let $(X, \mathcal{O}_X)$ be an algebraic variety and let $x \in X$. Prove that $\mathcal{O}_{X,x}$ is a local ring with maximal ideal $m = \{f \in \mathcal{O}_{X,x} \mid f(x) = 0\}$.
  (6) (★) Let $(X, \mathcal{O}_X)$ be an algebraic variety. Show that $\mathcal{O}_X$ is a coherent sheaf.
  (7) Show that the homomorphism

$$\varphi : (k[X_0, ..., X_n]_{FX_0})_0 \to k[X_1, ..., X_n]_{^\flat F}$$

  defined by $P/(FX_0)^r \mapsto (^\flat P)/(^\flat F)^r$ is an isomorphism of rings, where $F \in k[X_0, ..., X_n]$ is homogeneous. (Refer to the end of the notes from class on 2 March; this is a piece of the proof of III.8.4, to show that a projective variety is an algebraic variety.)
- Exercises A, 1

- Exercises B, 2 (added 3/3) Also, after doing this exercise, try the following two examples (added 14/3):
  (a) Let $R = k[X]$ and $M = R/(X^2)$. Write out the resolution of M.
  (b) Let $R = k[X, Y]$ and $M = k$. Write out the resolution of M.

Chapter IV Exercises. Due 19 April (Added 26/3)

(a) (★) Exercise IV, #1 on affine intersections (page 83)
(b) Exercise IV, #2 on projective intersections (page 84)
(c) (★) Give your own example which illustrates the dimension theorem (Theorem 3.7 on page 78-79).
(d) Appendix C: Problems, Problem I.1 (a,b,c,d,e) on page 216. This is about products of affine algebraic varieties.

Chapter V Exercises. Due 19 April (Added 26/3)

(a) (★) Prove: If V is an algebraic variety and $x \in V$ is a point, then x is non-singular in V if and only if the local ring $\mathcal{O}_{V,x}$ is regular.
(b) (★) Exercise #2, choose one or two of the examples (page 98)
(c) Exercise #7 (page 99)

Chapter VI Exercises. Due 4 May (Added 18/4, tentative list)

(a) (★) Let $Z = \{P\}$ where $P = (0, 0) \in k^2$. Give a finite scheme structure $(Z, \mathcal{O}_Z)$ that is not an algebraic variety.
(b) Prove the exercises about the $\flat$ and $\sharp$ operators from class. In particular, show that if $F, G \in k[X, Y, T]$ have no common components, then $F_\flat, G_\flat \in k[X, Y]$ have no common components.
(c) (★) Explicitly compute the intersection multiplicity $\mu_P(F, G)$ when $P = (1, 0)$, $F = Y$, and $G = X^2 + Y^2 - 1$.
(d) (★) Give examples to show why all the assumptions of Bézout's theorem are necessary for the statement.
(e) (★) In the affine plane, we know that the lines $X = 0$ and $X = 1$ do not intersect. In $\mathbb{P}^2$ show how the corresponding projective lines $X = 0$ and $X = T$ intersect at a point (by changing the line at infinity).
(f) (★) Show that Bézout's theorem holds for the curves $F = X^2 - 2XT + Y^2$ and $G = XT - Y^2$.
(g) Show that Bézout's theorem holds for the curves $F = TY - X^2$ and $G = T^2Y - X^3$.
(h) Challenging bonus project to think about: Investigate intersecting the trefoil and quadrifoil with regards to Bézout's theorem.

A few more exercises. Due 4 May (Added 29/4)

(a) Let X be a topological space with open cover $\mathcal{U} = \{U_0, U_1, U_2\}$, and let $\mathcal{F}$ be a sheaf of abelian groups on X. Examine the Cech complex in this case: write out the differentials and show that $C(\mathcal{U}, \mathcal{F})$ is a complex.
(b) Let X be a projective algebraic variety and $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be an exact sequence of coherent sheaves on X. Prove $\chi(\mathcal{F}) = \chi(\mathcal{F}') = \chi(\mathcal{F}'')$, where $\chi(\mathcal{F}) = \sum_{i=0}^{n} (-1)^i \dim_k H^i(X, \mathcal{F})$ is the Euler-Poincaré characteristic.
(c) Describe the structure sheaf of the scheme $\operatorname{Spec} k \times k$.
(d) Describe the structure sheaf of the scheme $\operatorname{Spec} \mathbb{Z}_{(2)}$.
(e) Give an example of an affine scheme that is not an affine algebraic variety.