EXERCISES FOR MA8203 ALGEBRAIC GEOMETRY

UPDATED LAST: MARCH 14, 2021

The numbered exercises come from Daniel Perrin's textbook.

Chapter I Exercises. Due 1 February

- 1
- 2
- 7, assume k is infinite
- 8

Chapter II Exercises. Due 1 March

- 1
- 5: you can use Macaulay2 to help find the resolution in part (c)

Chapter III Exercises. Due 15 March

- Exercises about sheaves from class.
  (1) Let $\mathcal{F}$ be a sheaf on a topological space X and let $U \subseteq X$ be open. Show that $\mathcal{F}|_U$ is a sheaf on U.
  (2) Let $\pi : X \to Y$ be a continuous map of topological spaces and let $\mathcal{F}$ be a sheaf on X. Show that $\pi_*\mathcal{F}$, defined by $\pi_*\mathcal{F}(V) = \mathcal{F}(\pi^{-1}(V))$, is a sheaf on Y.
  (3) Let $\mathcal{F}$ be a sheaf of rings on a topological space X.
    (a) If $p \in X$, show that $\mathcal{F}_p$ is a ring.
    (b) If $(X, \mathcal{O}_X)$ is a ringed space, and $\mathcal{F}$ is an $\mathcal{O}_X$-module, show that $\mathcal{F}_p$ is an $\mathcal{O}_{X,p}$-module.
  (4) Show that a sequence $0 \to \mathcal{F} \to \mathcal{G}$ of sheaves of abelian groups on X is exact if and only if $0 \to \mathcal{F}(U) \to \mathcal{G}(U)$ is exact for every open set $U \subseteq X$. Bonus: give an example to show a similar statement is not true for surjections. (corrected 14/3)
- Exercises about algebraic varieties from class. (added 3/3)
  (5) Let $(X, \mathcal{O}_X)$ be an algebraic variety and let $x \in X$. Prove that $\mathcal{O}_{X,x}$ is a local ring with maximal ideal $m = \{f \in \mathcal{O}_{X,x} \mid f(x) = 0\}$.
  (6) Let $(X, \mathcal{O}_X)$ be an algebraic variety. Show that $\mathcal{O}_X$ is a coherent sheaf.
  (7) Show that the homomorphism

  $$\varphi : (k[X_0, ..., X_n]_{FX_0})_0 \to k[X_1, ..., X_n]_{^\flat F}$$

  defined by $P/(FX_0)^r \mapsto (^\flat P)/(^\flat F)^r$ is an isomorphism of rings, where $F \in k[X_0, ..., X_n]$ is homogeneous. (Refer to the end of the notes from class on 2 March; this is a piece of the proof of III.8.4, to show that a projective variety is an algebraic variety.)
- Exercises A, 1
- Exercises B, 2 (added 3/3) Also, after doing this exercise, try the following two examples (added 14/3):

(a) Let $R = k[X]$ and $M = R/(X^2)$. Write out the resolution of M.
(b) Let $R = k[X, Y]$ and $M = k$. Write out the resolution of M.