

EXERCISES FOR MA8203 ALGEBRAIC GEOMETRY

UPDATED LAST: MAY 7, 2021

The numbered exercises come from Daniel Perrin's textbook.

(★) indicates exercise that may be asked on final exam (added 21/4)

Chapter I Exercises. Due 1 February

- (★) 1
- (★) 2
- (★) 7, assume k is infinite
- 8

Chapter II Exercises. Due 1 March

- (★) 1
- (★) 5: you can use Macaulay2 to help find the resolution in part (c)

Chapter III Exercises. Due 15 March

- Exercises about sheaves from class.
 - (1) (★) Let \mathcal{F} be a sheaf on a topological space X and let $U \subseteq X$ be open. Show that $\mathcal{F}|_U$ is a sheaf on U .
 - (2) (★) Let $\pi : X \rightarrow Y$ be a continuous map of topological spaces and let \mathcal{F} be a sheaf on X . Show that $\pi_*\mathcal{F}$, defined by $\pi_*\mathcal{F}(V) = \mathcal{F}(\pi^{-1}(V))$, is a sheaf on Y .
 - (3) (★) Let \mathcal{F} be a sheaf of rings on a topological space X .
 - (a) If $p \in X$, show that \mathcal{F}_p is a ring.
 - (b) If (X, \mathcal{O}_X) is a ringed space, and \mathcal{F} is an \mathcal{O}_X -module, show that \mathcal{F}_p is an $\mathcal{O}_{X,p}$ -module.
 - (4) (★) Show that a sequence $0 \rightarrow \mathcal{F} \rightarrow \mathcal{G}$ of sheaves of abelian groups on X is exact if and only if $0 \rightarrow \mathcal{F}(U) \rightarrow \mathcal{G}(U)$ is exact for every open set $U \subseteq X$. Bonus: give an example to show a similar statement is not true for surjections. (corrected 14/3)
- Exercises about algebraic varieties from class. (added 3/3)
 - (5) (★) Let (X, \mathcal{O}_X) be an algebraic variety and let $x \in X$. Prove that $\mathcal{O}_{X,x}$ is a local ring with maximal ideal $m = \{f \in \mathcal{O}_{X,x} \mid f(x) = 0\}$.
 - (6) (★) Let (X, \mathcal{O}_X) be an algebraic variety. Show that \mathcal{O}_X is a coherent sheaf.
 - (7) Show that the homomorphism

$$\varphi : (k[X_0, \dots, X_n]_{FX_0})_0 \rightarrow k[X_1, \dots, X_n]_{\flat F}$$

defined by $P/(FX_0)^r \mapsto (\flat P)/(\flat F)^r$ is an isomorphism of rings, where $F \in k[X_0, \dots, X_n]$ is homogeneous. (Refer to the end of the notes from class on 2 March; this is a piece of the proof of III.8.4, to show that a projective variety is an algebraic variety.)

- Exercises A, 1

- Exercises B, 2 (added 3/3) Also, after doing this exercise, try the following two examples (added 14/3):
 - (a) Let $R = k[X]$ and $M = R/(X^2)$. Write out the resolution of M .
 - (b) Let $R = k[X, Y]$ and $M = k$. Write out the resolution of M .

Chapter IV Exercises. Due 19 April (Added 26/3)

- (a) (★) Exercise IV, #1 on affine intersections (page 83)
- (b) Exercise IV, #2 on projective intersections (page 84)
- (c) (★) Give your own example which illustrates the dimension theorem (Theorem 3.7 on page 78-79).
- (d) Appendix C: Problems, Problem I.1 (a,b,c,d,e) on page 216. This is about products of affine algebraic varieties.

Chapter V Exercises. Due 19 April (Added 26/3)

- (a) (★) Prove: If V is an algebraic variety and $x \in V$ is a point, then x is non-singular in V if and only if the local ring $\mathcal{O}_{V,x}$ is regular.
- (b) (★) Exercise #2, choose one or two of the examples (page 98)
- (c) Exercise #7 (page 99)

Chapter VI Exercises. Due 4 May (Added 18/4, tentative list)

- (a) (★) Let $Z = \{P\}$ where $P = (0, 0) \in k^2$. Give a finite scheme structure (Z, \mathcal{O}_Z) that is not an algebraic variety.
- (b) Prove the exercises about the \flat and \sharp operators from class. In particular, show that if $F, G \in k[X, Y, T]$ have no common components, then $F_\flat, G_\flat \in k[X, Y]$ have no common components.
- (c) (★) Explicitly compute the intersection multiplicity $\mu_P(F, G)$ when $P = (1, 0)$, $F = Y$, and $G = X^2 + Y^2 - 1$.
- (d) (★) Give examples to show why all the assumptions of Bézout's theorem are necessary for the statement.
- (e) (★) In the affine plane, we know that the lines $X = 0$ and $X = 1$ do not intersect. In \mathbb{P}^2 show how the corresponding projective lines $X = 0$ and $X = T$ intersect at a point (by changing the line at infinity).
- (f) (★) Show that Bézout's theorem holds for the curves $F = X^2 - 2XT + Y^2$ and $G = XT - Y^2$.
- (g) Show that Bézout's theorem holds for the curves $F = TY - X^2$ and $G = T^2Y - X^3$.
- (h) Challenging bonus project to think about: Investigate intersecting the trefoil and quadrifoil with regards to Bézout's theorem.

A few more exercises. Due 4 May (Added 29/4)

- (a) (★) Let X be a topological space with open cover $\mathcal{U} = \{U_0, U_1, U_2\}$, and let \mathcal{F} be a sheaf of abelian groups on X . Examine the Čech complex in this case: write out the differentials and show that $C(\mathcal{U}, \mathcal{F})$ is a complex.
- (b) Let X be a projective algebraic variety and $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$ be an exact sequence of coherent sheaves on X . Prove $\chi(\mathcal{F}) = \chi(\mathcal{F}') = \chi(\mathcal{F}'')$, where $\chi(\mathcal{F}) = \sum_{i=0}^n (-1)^i \dim_k H^i(X, \mathcal{F})$ is the Euler-Poincaré characteristic.
- (c) (★) Describe the structure sheaf of the scheme $\text{Spec } k \times k$.
- (d) (★) Describe the structure sheaf of the scheme $\text{Spec } \mathbb{Z}_{(2)}$.

- (e) (★) Give an example of an affine scheme that is not an affine algebraic variety.