EXERCISES

1. Homework 1: "Due" 8th February

Problem 1.1. Let $\mathcal{F}$ be a sheaf on a topological space X. Suppose that for every $\mathfrak{p} \in X$, $\mathcal{F}_{\mathfrak{p}} = 0$. Show that $\mathcal{F} = 0$.

Problem 1.2. Let $f \colon \mathcal{F} \to \mathcal{G}$ be a homomorphism of sheaves over X. (*Hint: use the previous problem*)

a) Show that the following are equivalent. This is the definition of an injective sheaf.
   i) $\ker f = 0$
   ii) f_U is injective for every open sets $U \subseteq X$
   iii) $f_{\mathfrak{p}} \colon \mathcal{F}_{\mathfrak{p}} \to \mathcal{G}_{\mathfrak{p}}$ is injective for every $\mathfrak{p} \in X$
b) Show that the following are equivalent. This is the definition of a surjective sheaf.
   i) $\operatorname{coker} f = 0$
   ii) $f_{\mathfrak{p}} \colon \mathcal{F}_{\mathfrak{p}} \to \mathcal{G}_{\mathfrak{p}}$ is surjective for every $\mathfrak{p} \in X$.
c) Show that the following are equivalent. This is the definition of a sheaf isomorphism.
   i) $\ker f = \operatorname{coker} f = 0$
   ii) f_U is an isomorphism for every open sets $U \subseteq X$
   iii) $f_{\mathfrak{p}} \colon \mathcal{F}_{\mathfrak{p}} \to \mathcal{G}_{\mathfrak{p}}$ is an isomorphism for every $\mathfrak{p} \in X$.
   iv) There exists a sheaf homomorphism $g \colon \mathcal{G} \to \mathcal{F}$ such that $g_U \circ f_U$ and $f_u \circ g_U$ are the identity on $\mathcal{G}(U)$ and $\mathcal{F}(U)$ respectively for all open sets $U \subseteq X$.

Problem 1.3. Give an example of a space X and a sheaf homomorphism $f \colon \mathcal{F} \to \mathcal{G}$ over X such that f is surjective but f_U is not surjective for some open set $U \subseteq X$. (*Hint:* Use the following constructions. Let G be an Abelian group. 1) Assign every open set of $\mathbb{R}$ to the group G. 2) Take a point $\mathfrak{p} \in X$. For any open set $U \subseteq \mathbb{R}$, assign G to U if $\mathfrak{p} \in U$. Otherwise assign 0 to U. This is called the sky scraper sheaf of G at $\mathfrak{p}$. 3) The direct sum of sheaves is a sheaf.)

Problem 1.4. Let R be a commutative ring, and $\mathfrak{p} \subseteq R$ a prime. Then

$$k(\mathfrak{p}) = \frac{R_{\mathfrak{p}}}{\mathfrak{p}R_{\mathfrak{p}}}$$

is isomorphic to the field of fractions of $R/\mathfrak{p}$.

Problem 1.5. If $I, J \subseteq R$ are ideals, then $V(I) = V(J)$ if and only if $\sqrt{I} = \sqrt{J}$.

Problem 1.6. Show that $D(x_1, \ldots, x_n) \subseteq \operatorname{spec} R$ is compact.

2. Homework 2: "Due" 22 February

Problem 2.1. Let $R = k[x_1, \ldots, x_n]$ with k algebraically closed.

a) Prove the that the following forms of the Nullstellsatz are equivalent.
   i) $\mathcal{I}$ and $\mathcal{Z}$ give a bijection between the algebraic sets of k^n and radical ideals of R.
   ii) For every proper ideal $I \subseteq R$, we have

$$\sqrt{I} = \bigcap_{\mathfrak{m} \in V(I) \cap \operatorname{MaxSpec} R} \mathfrak{m}$$

b) Use Lemma 4.8 to show that ii) holds in R. This provides a second proof of the nullstellensatz.

Problem 2.2. Assuming k is algebraically closed, show that every point in $\mathbb{P}^n_k$ is an algebraic set.

Problem 2.3. For an algebraically closed field k, show that

$$\mathcal{O}_{\mathbb{P}^n_k}(\mathbb{P}^n_k) = k$$

Problem 2.4. Proved that every morphism of projective varieties is regular.

Problem 2.5. Let k be an algebraically closed field.

a) Suppose $I \subseteq k[x_1, \ldots, x_n]$ and $J \subseteq k[y_1, \ldots, y_n]$ are ideals such that
$$\frac{k[x_1, \ldots, x_n]}{\sqrt{I}} \cong \frac{k[y_1, \ldots, y_m]}{\sqrt{J}}.$$
Show that varieties $V(I)$ and $V(J)$ are isomorphic.

b) Conclude that for an affine variety V, and an open set $U \subseteq V$, the definition of a regular function
$$\sigma \colon U \to k$$
does not depend on the choice on the presentation of the coordinate ring of V.

c) Formulate the corresponding statement for quasi-projective varieties.

Problem 2.6. Let V be the image of the following map.
$$\varphi \colon \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^3_{\mathbb{C}} \qquad \varphi(x_0, x_1) = (x_0^3, x_0^2 x_1, x_0 x_1^2, x_1^3)$$

(1) Show that
$$V = V(y_0 y_3 - y_1 y_2, y_1^2 - y_0 y_2, y_2^2 - y_1 y_3).$$
(*Hint*: Restrict to the open affine sets $y_0 \neq 0$ and $y_3 \neq 0$.)

(2) Let
$$\psi \colon \mathbb{P}^1_{\mathbb{C}} \to V \qquad \text{and} \qquad \iota \colon V \to \mathbb{P}^3_{\mathbb{C}}$$
respectively be the corestriction of φ and the natural inclusion. Compute the corresponding maps on coordinate rings
$$\hat{\psi} \colon \mathbb{C}[V] \to \mathbb{C}[\mathbb{P}^1_{\mathbb{C}}] \qquad \hat{\varphi} \colon \mathbb{C}[\mathbb{P}^3_{\mathbb{C}}] \to \mathbb{C}[\mathbb{P}^1_{\mathbb{C}}] \qquad \hat{\iota} \colon \mathbb{C}[\mathbb{P}^3_{\mathbb{C}}] \to \mathbb{C}[V].$$

(3) Show that ψ is a homeomorphism, but not an isomorphism of varieties.

Problem 2.7 (Optional: for the benefit of the motivated student and will not be on the final exam)**.** Let R be a commutative ring. Let $U \subseteq \operatorname{spec} R$ be an open set. A function
$$\sigma \colon U \to \bigsqcup R_{\mathfrak{p}}$$
is regular, if there is an open cover $U = \bigcup U_i$ and elements $f_i, g_{\in} R$ with $U_i \subseteq D(g_i)$ such that for every $\mathfrak{p} \in U_i$,
$$\sigma(\mathfrak{p}) = \frac{f_i}{g_i} \in R_{\mathfrak{p}}.$$
Let $\mathcal{O}_R(U)$ denote the collection of regular functions on U.

A sheaf which is isomorphic to the pair $(\operatorname{spec} R, \mathcal{O}_R)$ is called an affine scheme. A scheme is a sheaf $\mathcal{O}$ on a space X such that there is an open cover $U = \bigcup U_i$ where $(U_i, \mathcal{O} \mid_{U_i})$ is an affine scheme.

Show the following.

a) $\mathcal{O}_R$ is a sheaf of rings
b) $\mathcal{O}_R(\operatorname{spec} R) = R$
c) $\mathcal{O}(D(g)) = R_g$
d) For any $\mathfrak{p} \in \operatorname{spec} R$, $\mathcal{O}_{R,\mathfrak{p}} = R_{\mathfrak{p}}$.
e) Any affine variety over an algebraically closed field is an affine scheme.
f) Any variety over an algebraically closed field is scheme.

3. Homework 3: "Due" 8 March

Let k be an algebraically closed field.

Problem 3.1. Let G be the group of automorphisms of $\mathbb{A}^n_k$in the category of varieties. Show that G is not isomorphic to $\mathrm{GL}(n, k)$, the group of $n \times n$ invertible matrices over k.

Problem 3.2. Let $\varphi \colon V \to W$ be a morphism of varieties. For every $\mathfrak{p} \in V$, construct a map between the tangent spaces
$$\mathcal{T}_{V,\mathfrak{p}} \to \mathcal{T}_{W,\varphi(\mathfrak{p})}.$$
(*Hint*: For every open $U \subseteq V$, construct a map $\mathcal{T}_V(\varphi^{-1}(U)) \to \mathcal{T}_W(U)$.)

Problem 3.3. Consider the polynomials in $k[w,x,y,z]$.

$$f = x^2 - xz - yw \qquad g = yz - xw - zw$$

(1) Is the affine variety $V(f,g) \subseteq \mathbb{A}^4_k$ nonsingular?
(2) Is the projective variety $V^+(f,g) \subseteq \mathbb{P}^3_k$ nonsingular? This variety is the elliptic quadric. (*Hint*: Consider the points $\mathfrak{p} = [1:0:0:0] \in \mathbb{P}^3_k$ and $\mathfrak{q} = [1:0:-1] \in \mathbb{P}^2_k$. Show that the variety $V^+(f,g)\backslash\mathfrak{p}$ is isomorphic to $V^+(y^2z - x^3 + xz^2)\backslash\mathfrak{q} \subseteq \mathbb{P}^2_k$)

Problem 3.4. Let V be a projective variety over k. Let S be the singular points of V. Show that S is closed. (*Hint*: Use the Jacobian criterion.)

Problem 3.5. Let $f = y^2z - x^3 \in k[x,yz]$. Let V be curve in $\mathbb{P}^2$ defined by f. Find the singular point of V and compute its tangent space. The variety V is called a cuspidal cubic. Can you justify this name?

Problem 3.6. Recall that an R-module is free of rank n if it is of the form R^n. Let V be a projective variety over k and $\mathcal{F}$ a coherent sheaf on V. Let $\mathcal{O}_V^n$ be the direct sum of $\mathcal{O}_v$ with itself n-times.

(1) Show that the following are equivalent.
  (a) The stalk $\mathcal{F}_\mathfrak{p}$ are free $\mathcal{O}_{V,\mathfrak{p}}$-module for every $\mathfrak{p} \in V$
  (b) There is an open cover $V = \bigcup U_i$ such that for each i,
  $$\mathcal{F}\mid_{U_i} \cong \mathcal{O}_V^n \mid_{U_i}.$$
  This is the definition of a locally free sheaf or a vector bundle.
(2) Suppose $\mathcal{F}$ be a locally free. Show that if V is connected, then the rank of the free $\mathcal{O}_{V,\mathfrak{p}}$-module $\mathcal{F}_\mathfrak{p}$ is the same for all $\mathfrak{p} \in V$. This is number is called the rank of $\mathcal{F}$.

Problem 3.7 (Optional: for the benefit of the motivated student and will not be on the final exam)**.** Recall from Problem 2.7 the definition of an affine scheme and a scheme.

(1) Let R be a commutative ring, and let a M be an R-module. Show that there is a unique sheaf $\tilde{M}$ on $\operatorname{spec} R$ such that
  (a) $\tilde{M}$ an $\mathcal{O}_R$-module
  (b) $\tilde{M}(D(g)) = M_g$ for every nonzero $g \in R$
  (c) $\tilde{M}_\mathfrak{p} = M_\mathfrak{p}$ for every $\mathfrak{p} \in \operatorname{spec} R$.
  (*Hint*: Use the same construction in Problem 2.7.)
(2) For R-modules M and N, show that every $\mathcal{O}_R$-module homomorphism
  $$\varphi\colon \tilde{M} \to \tilde{N}$$
  is of the form $\varphi = \tilde{f}$ for an R-module homomorphism $f\colon M \to N$.
(3) For an affine scheme $(\operatorname{spec} R, \mathcal{O}_R)$, a sheaf $\mathcal{F}$ on $\operatorname{spec} R$ is quasi-coherent if $\mathcal{F} = \tilde{M}$ for some R-module. Furthermore $\mathcal{F}$ is coherent if we can choose M to be finitely generated. Prove the the category of quasi-coherent sheaves (whose morphisms are $\mathcal{O}_R$-module homomorphisms) is equivalent to the category of R-modules.
(4) Let $(X, \mathcal{O})$ be a scheme. Show that the following are equivalent for a sheaf $\mathcal{F}$ of $\mathcal{O}$-modules.
  (a) There exists an open cover $X = \cup U_i$ such that $(U_i, \mathcal{O}|_{U_i}) \cong (\operatorname{spec} R_i, \mathcal{O}_{R_i})$ is an affine scheme, and $\mathcal{F}|_{U_i}$ is coherent, i.e. $\mathcal{F}|_{U_i} \cong \tilde{M}_i$ for an R_i-module M_i.
  (b) For every open cover $X = \cup U_i$ such that $(U_i, \mathcal{O}|_{U_i}) \cong (\operatorname{spec} R_i, \mathcal{O}_{R_i})$ is an affine scheme, the restriction $\mathcal{F}|_{U_i}$ is coherent, i.e. $\mathcal{F}|_{U_i} \cong \tilde{M}_i$ for an R_i-module M_i.
  This is the definition of a quasi-coherent sheaf over X. Moreover, the exercise is still true if we replace quasi-coherent with coherent.
(5) Let V be a projective variety. Then V is a scheme. If $\mathcal{F}$ is a coherent sheaf in the sense defined in class, show that it is coherent in the above sense. **Hard**: Prove the converse, i.e. that both senses of coherent shaves are the same. Note, this should give you an appreciation for the niceness of our situation. Coherent sheaves can get pretty crazy, but in our situation describing them with graded modules makes life much easier.

4. Homework 4: "Due" 22 March

Problem 4.1.

(1) Let $\varphi\colon : X \to Y$ be a continuous map between topological spaces. Let $\mathcal{F}$ be a sheaf on X, and for any open set $U \subseteq Y$, define $\varphi_*\mathcal{F}(U) = \mathcal{F}(\varphi^{-1}(U))$. Show that $\varphi_*\mathcal{F}$ is a sheaf on Y.

(2) Let V and W be varieties, and $\varphi\colon V \to W$ be a morphism of varieties. If $\mathcal{M}$ is an $\mathcal{O}_V$-module, show that $\varphi_*\mathcal{M}$ is an $\mathcal{O}_W$-module. (*Hint*: There is a sheaf homomorphism $\mathcal{O}_W \to \varphi_*\mathcal{O}_V$.)

(3) Let V and W be projective varieties, and let R and S be their respective homogenous coordinate rings. Let $\varphi\colon V \to W$ be a morphism of varieties, and let $\hat{\varphi}\colon S \to R$ be the associated map on the coordinate rings. Let M be an a graded$R$-module. Now M is also an S-module. Let M_R and M_S denote the same module with different actions. Prove that

$$\varphi_*\tilde{M}_R = \tilde{M}_S.$$

Thus we have a functor $\varphi_*\colon \mathrm{Qcoh}V \to \mathrm{Qcoh}W$.

(4) In the previous exercise, if $\mathcal{M}$ is a coherent sheaf over V, is $\varphi_*\mathcal{M}$ coherent over W?

Problem 4.2. Let X be a space, and $\mathcal{F}$ a sheaf on X. We say that $\mathcal{F}$ is flasque if $U' \subseteq U \subseteq X$ are open sets implies $|^U_{U'}\colon \mathcal{F}(U) \to \mathcal{F}(U')$ is surjective. Suppose further that

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{K} \to 0$$

is an exact sequence of sheaves with $\mathcal{F}$ flasque.

(1) Show that for every open set $U \subseteq X$, the sequence

$$0 \to \mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{K}(U) \to 0$$

is exact.

(2) Show that if $\mathcal{G}$ is also flasque, the $\mathcal{K}$ must be flasque too.

(3) Let $\varphi\colon X \to Y$ be a continuous map to another space. Show that $\varphi_*\mathcal{F}$ is also flasque.

Problem 4.3. Let V be a variety.

(1) Show that every injective $\mathcal{O}_V$-module is a flasque sheaf.

(2) Prove that $H^i(\mathcal{F}) = 0$ for every flasque $\mathcal{O}_V$-module $\mathcal{F}$.

(3) Let $\mathcal{M}$ be a $\mathcal{O}_V$-module and let

$$0 \to \mathcal{M} \to \mathcal{F}^0 \xrightarrow{\partial^0} \mathcal{F}^1 \xrightarrow{\partial^1} \mathcal{F}^2 \xrightarrow{\partial^2} \cdots$$

be a flasque resolution of $\mathcal{M}$. This means that

(a) each $\mathcal{F}^i$ is flasque
(b) $\ker \partial^0 = \mathcal{M}$
(c) $\ker \partial^i = \mathrm{Im}\, \partial^{i-1}$ for all $i > 0$

Prove that for all $i \geq 0$

$$H^i(\mathcal{M}) = H^i(\mathcal{F}^\bullet) = \ker \partial^i_V / \mathrm{Im}\, \partial^{i-1}_V$$

where $\partial^{-1} = 0$. In short, we can use flasque resolution to compute sheaf cohomology.

(4) Note that the solutions (should) work in the category of sheaves; see Remark 11.9 from lecture). Conclude that sheaf cohomology is the same regardless if we compute it in the category of of sheaves or $\mathcal{O}_V$-modules.

(5) Let V and W be varieties and $\varphi\colon V \to W$ a morphism. Assume that there exists a affine cover $W = \bigcup U_i$ such that $\varphi^{-1}(U_i)$ is also affine (such morphisms are called affine, and if V and W are either both affine or projective, the assumption is automatically satisfied). If $\mathcal{M}$ is an $\mathcal{O}_V$-module, show that

$$H^i(\varphi_*\mathcal{M}) = H^i(\mathcal{M})$$

for all i.

(6) The following result is due to is called Noether Normalization: Let R be a finitely generated graded k-algebra with $\dim R = d$. Then there exists algebraically independent homogenous elements $f_1, \ldots, f_d \in R$.

Use this result and our previous work to prove the following special case of a theorem of Grothendieck. Let V be a projective variety. If $\mathcal{M}$ is a quasi-coherent sheaf on V, then $H^i(\mathcal{M}) = 0$ for all $i > \dim V$.

Problem 4.4. Let $V = V^+(f) \subseteq \mathbb{P}^2_{\mathbb{C}}$ where $f = x_0^n + x_1^n + x_2^n \in \mathbb{C}[x_0, x_1.x_2]$. Prove that $V \cap D^+(x_1), V \cap D^+(x_2)$ are an open cover of V. Use Čech cohomology to prove that

$$\dim_{\mathbb{C}} H^1(\mathcal{O}_V) = \frac{1}{2}(n-1)(n-2).$$

Problem 4.5 (Optional: for the benefit of the motivated student and will not be on the final exam)**.** Let $X = \operatorname{spec} R$ be an affine scheme. Recall the definition of quasi-coherent sheaf over X from last week.

1. $\Gamma(X,)$ and $\tilde{}$ give an equivalence of categories between affine quasi-coherent sheaves and R-modules.
2. Conclude that for any quasi-coherent sheaf $\mathcal{M}$, the sheaf cohomology $H^i(\mathcal{M})$ vanishes for all $i > 0$.

5. Homework 5

Problem 5.1. Let k be an algebraically closed field, and let $V \subseteq \mathbb{P}^d_k$ be a projective hypersurface. This means that $V = V^+(f)$ with homogenous $f \in k[x_0, \dots, x_d]$ and the ideal (f) is radical. Compute

$$H^i(\mathcal{O}_V(n))$$

for all $i, n \in \mathbb{Z}$. (You do not need to be too specific about $H^{d-1}((_V(n)))$

Problem 5.2. Consider Example 13.8 from class: let $V = V^+(y^2z - x^3 + xz^2) \subseteq \mathbb{P}^2_k$ with char $k \neq 2$. Set

$$f = \frac{ax + by + cz}{z} \in K(V).$$

Rigoresly show that

$$(f) = \mathfrak{p} + \mathfrak{q} + \mathfrak{r} - 3[0:0:1]$$

for some not necessarily distinct closed points $\mathfrak{p}, \mathfrak{q}, \mathfrak{r} \in V^c$., (*Hint*: Define a map $\psi \colon k[x_0, x_1] \to k[V]$ by $x_0 \mapsto ax + by + cz$ and $X_1 \mapsto z$. This Defines a morphism $\varphi \colon V \to \mathbb{P}^1_k$. Show that $[K(V) : K(\mathbb{P}^1_k) = 3$. Then, use the claim in the proof of Theorem 13.7 from class.)

Problem 5.3. Let V be an affine curve over an algebraically closed field k. Let $R = k[V]$ be the coordinate ring. Show that $\operatorname{Cl} V = 0$ if and only if R is a UFD. (*Hint*: Use may use the theorem that a Noetherian domain R is a UFD if and only if every prime $\mathfrak{p} \in \operatorname{spec} R$ with $\operatorname{ht} \mathfrak{p} = 1$ is principal.)

Problem 5.4. Let k be an algebraically closed field.

1. Show that for each $n \in \mathbb{Z}$, $\mathcal{O}_{\mathbb{P}^1_k}(n)$ is an invertible sheaf.
2. Compute the Cartier divisor $D(\mathcal{O}_{\mathbb{P}^1_k}(n))$.
3. What is $\deg D(\mathcal{O}_{\mathbb{P}^1_k}(n))$? (Recall that $\operatorname{Pic} \mathbb{P}^1_k \cong \operatorname{Cl} \mathbb{P}^1_k$, and so the degree of a Cartier divisor is well defined.)
4. Conclude that every invertible sheaf over $\mathbb{P}^1_k$ is isomorphic to $_{\mathbb{P}^1_k}(n)$ for some $n \in \Omega$.

6. Homework 6

Problem 6.1. For a curve V over an algebraically closed field k, show that V is birational to $\mathbb{P}^1_k$ if and only if $K(V)$ is a purely transcendental extension of k.

Problem 6.2. Let $V = V(y^3 - x^2) \subseteq \mathbb{A}^2_{\mathbb{C}}$. Show that V is singular and find a nonsingular curve which is birational to V. Sketch the curve.

Problem 6.3. Let V be a projective variety over $\mathbb{C}$. Suppose that every invertible sheaf over V is isomorphic to $\mathcal{O}_V(n)$ for some $n \in \mathbb{Z}$. Show that V is birational to $\mathbb{P}^1_{\mathbb{C}}$. (*Hint*: use the fact that $\mathbb{C}$ is uncountable.)

Problem 6.4. Let $V = V^+(f) \subseteq \mathcal{P}^2$ be hypersurface (see Problem 5.1) with $\deg f = d$. Show that

$$p_a(V) = \frac{1}{2}(d-1)(d-2).$$

Compare this problem with Problem 4.4.

Problem 6.5. Let $V \subseteq \mathcal{P}^2$ be an elliptic curve and a hypersurface (see Problem 5.1). Show that the closed points V^c has the structure of an Abelian group. (*Hint*: Generalize Problem 5.2 to this general context, and then use the logic in Example 13.8 from class.)