EXERCISES

1. Homework 1: "Due" 8th February

Problem 1.1. Let $\mathcal{F}$ be a sheaf on a topological space X. Suppose that for every $\mathfrak{p} \in X$, $\mathcal{F}_{\mathfrak{p}} = 0$. Show that $\mathcal{F} = 0$.

Problem 1.2. Let $f\colon \mathcal{F} \to \mathcal{G}$ be a homomorphism of sheaves over X. (*Hint: use the previous problem*)

a) Show that the following are equivalent. This is the definition of an injective sheaf.
   i) $\ker f = 0$
   ii) f_U is injective for every open sets $U \subseteq X$
   iii) $f_{\mathfrak{p}}\colon \mathcal{F}_{\mathfrak{p}} \to \mathcal{G}_{\mathfrak{p}}$ is injective for every $\mathfrak{p} \in X$

b) Show that the following are equivalent. This is the definition of a surjective sheaf.
   i) $\operatorname{coker} f = 0$
   ii) $f_{\mathfrak{p}}\colon \mathcal{F}_{\mathfrak{p}} \to \mathcal{G}_{\mathfrak{p}}$ is surjective for every $\mathfrak{p} \in X$.

c) Show that the following are equivalent. This is the definition of a sheaf isomorphism.
   i) $\ker f = \operatorname{coker} f = 0$
   ii) f_U is an isomorphism for every open sets $U \subseteq X$
   iii) $f_{\mathfrak{p}}\colon \mathcal{F}_{\mathfrak{p}} \to \mathcal{G}_{\mathfrak{p}}$ is an isomorphism for every $\mathfrak{p} \in X$.
   iv) There exists a sheaf homomorphism $g\colon \mathcal{G} \to \mathcal{F}$ such that $g_U \circ f_U$ and $f_u \circ g_U$ are the identity on $\mathcal{G}(U)$ and $\mathcal{F}(U)$ respectively for all open sets $U \subseteq X$.

Problem 1.3. Give an example of a space X and a sheaf homomorphism $f\colon \mathcal{F} \to \mathcal{G}$ over X such that f is surjective but f_U is not surjective for some open set $U \subseteq X$. (*Hint:* Use the following constructions. Let G be an Abelian group. 1) Assign every open set of $\mathbb{R}$ to the group G. 2) Take a point $\mathfrak{p} \in X$. For any open set $U \subseteq \mathbb{R}$, assign G to U if $\mathfrak{p} \in U$. Otherwise assign 0 to U. This is called the sky scraper sheaf of G at $\mathfrak{p}$. 3) The direct sum of sheaves is a sheaf.)

Problem 1.4. Let R be a commutative ring, and $\mathfrak{p} \subseteq R$ a prime. Then

$$k(\mathfrak{p}) = \frac{R_{\mathfrak{p}}}{\mathfrak{p}R_{\mathfrak{p}}}$$

is isomorphic to the field of fractions of $R/\mathfrak{p}$.

Problem 1.5. If $I, J \subseteq R$ are ideals, then $V(I) = V(J)$ if and only if $\sqrt{I} = \sqrt{J}$.

Problem 1.6. Show that $D(x_1, \ldots, x_n) \subseteq \operatorname{spec} R$ is compact.

2. Homework 2: "Due" 22 February

Problem 2.1. Let $R = k[x_1, \ldots, x_n]$ with k algebraically closed.

a) Prove the that the following forms of the Nullstellsatz are equivalent.
   i) $\mathcal{I}$ and $\mathcal{Z}$ give a bijection between the algebraic sets of k^n and radical ideals of R.
   ii) For every proper ideal $I \subseteq R$, we have

$$\sqrt{I} = \bigcap_{\mathfrak{m} \in V(I) \cap \operatorname{MaxSpec} R} \mathfrak{m}$$

b) Use Lemma 4.8 to show that ii) holds in R. This provides a second proof of the nullstellensatz.

Problem 2.2. Assuming k is algebraically closed, show that every point in $\mathbb{P}^n_k$ is an algebraic set.

Problem 2.3. For an algebraically closed field k, show that

$$\mathcal{O}_{\mathbb{P}^n_k}(\mathbb{P}^n_k) = k$$

Problem 2.4. Proved that every morphism of projective varieties is regular.

Problem 2.5. Let k be an algebraically closed field.

a) Suppose $I \subseteq k[x_1, \ldots, x_n]$ and $J \subseteq k[y_1, \ldots, y_n]$ are ideals such that
$$\frac{k[x_1, \ldots, x_n]}{\sqrt{I}} \cong \frac{k[y_1, \ldots, y_m]}{\sqrt{J}}.$$
Show that varieties $V(I)$ and $V(J)$ are isomorphic.

b) Conclude that for an affine variety V, and an open set $U \subseteq V$, the definition of a regular function
$$\sigma \colon U \to k$$
does not depend on the choice on the presentation of the coordinate ring of V.

c) Formulate the corresponding statement for quasi-projective varieties.

Problem 2.6. Let V be the image of the following map.
$$\varphi \colon \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^3_{\mathbb{C}} \qquad \varphi(x_0, x_1) = (x_0^3, x_0^2 x_1, x_0 x_1^2, x_1^3)$$

(1) Show that
$$V = V(y_0 y_3 - y_1 y_2, y_1^2 - y_0 y_2, y_2^2 - y_1 y_3).$$
(*Hint*: Restrict to the open affine sets $y_0 \neq 0$ and $y_3 \neq 0$.)

(2) Let
$$\psi \colon \mathbb{P}^1_{\mathbb{C}} \to V \qquad \text{and} \qquad \iota \colon V \to \mathbb{P}^3_{\mathbb{C}}$$
respectively be the corestriction of φ and the natural inclusion. Compute the corresponding maps on coordinate rings
$$\hat{\psi} \colon \mathbb{C}[V] \to \mathbb{C}[\mathbb{P}^1_{\mathbb{C}}] \qquad \hat{\varphi} \colon \mathbb{C}[\mathbb{P}^3_{\mathbb{C}}] \to \mathbb{C}[\mathbb{P}^1_{\mathbb{C}}] \qquad \hat{\iota} \colon \mathbb{C}[\mathbb{P}^3_{\mathbb{C}}] \to \mathbb{C}[V].$$

(3) Show that φ is a homeomorphism, but not an isomorphism of varieties.

Problem 2.7 (Optional: for the benefit of the motivated student and will not be on the final exam)**.** Let R be a commutative ring. Let $U \subseteq \operatorname{spec} R$ be an open set. A function
$$\sigma \colon U \to \bigsqcup R_{\mathfrak{p}}$$
is regular, if there is an open cover $U = \bigcup U_i$ and elements $f_i, g_{\in} R$ with $U_i \subseteq D(g_i)$ such that for every $\mathfrak{p} \in U_i$,
$$\sigma(\mathfrak{p}) = \frac{f_i}{g_i} \in R_{\mathfrak{p}}.$$
Let $\mathcal{O}_R(U)$ denote the collection of regular functions on U.

A sheaf which is isomorphic to the pair $(\operatorname{spec} R, \mathcal{O}_R)$ is called an affine scheme. A scheme is a sheaf $\mathcal{O}$ on a space X such that there is an open cover $U = \bigcup U_i$ where $(U_i, \mathcal{O}\mid_{U_i})$ is an affine scheme.

Show the following.

a) $\mathcal{O}_R$ is a sheaf of rings
b) $\mathcal{O}_R(\operatorname{spec} R) = R$
c) $\mathcal{O}(D(g)) = R_g$
d) For any $\mathfrak{p} \in \operatorname{spec} R$, $\mathcal{O}_{R,\mathfrak{p}} = R_{\mathfrak{p}}$.
e) Any affine variety over an algebraically closed field is an affine scheme.
f) Any variety over an algebraically closed field is scheme.