

EXERCISES

1. HOMEWORK 1: "DUE" 8TH FEBRUARY

Problem 1.1. Let \mathcal{F} be a sheaf on a topological space X . Suppose that for every $\mathfrak{p} \in X$, $\mathcal{F}_{\mathfrak{p}} = 0$. Show that $\mathcal{F} = 0$.

Problem 1.2. Let $f: \mathcal{F} \rightarrow \mathcal{G}$ be a homomorphism of sheaves over X . (*Hint: use the previous problem*)

- a) Show that the following are equivalent. This is the definition of an injective sheaf.
- i) $\ker f = 0$
 - ii) f_U is injective for every open sets $U \subseteq X$
 - iii) $f_{\mathfrak{p}}: \mathcal{F}_{\mathfrak{p}} \rightarrow \mathcal{G}_{\mathfrak{p}}$ is injective for every $\mathfrak{p} \in X$
- b) Show that the following are equivalent. This is the definition of a surjective sheaf.
- i) $\operatorname{coker} f = 0$
 - ii) $f_{\mathfrak{p}}: \mathcal{F}_{\mathfrak{p}} \rightarrow \mathcal{G}_{\mathfrak{p}}$ is surjective for every $\mathfrak{p} \in X$.
- c) Show that the following are equivalent. This is the definition of a sheaf isomorphism.
- i) $\ker f = \operatorname{coker} f = 0$
 - ii) f_U is an isomorphism for every open sets $U \subseteq X$
 - iii) $f_{\mathfrak{p}}: \mathcal{F}_{\mathfrak{p}} \rightarrow \mathcal{G}_{\mathfrak{p}}$ is an isomorphism for every $\mathfrak{p} \in X$.
 - iv) There exists a sheaf homomorphism $g: \mathcal{G} \rightarrow \mathcal{F}$ such that $g_U \circ f_U$ and $f_U \circ g_U$ are the identity on $\mathcal{G}(U)$ and $\mathcal{F}(U)$ respectively for all open sets $U \subseteq X$.

Problem 1.3. Give an example of a space X and a sheaf homomorphism $f: \mathcal{F} \rightarrow \mathcal{G}$ over X such that f is surjective but f_U is not surjective for some open set $U \subseteq X$. (*Hint: Use the following constructions. Let G be an Abelian group. 1) Assign every open set of \mathbb{R} to the group G . 2) Take a point $\mathfrak{p} \in X$. For any open set $U \subseteq \mathbb{R}$, assign G to U if $\mathfrak{p} \in U$. Otherwise assign 0 to U . This is called the sky scraper sheaf of G at \mathfrak{p} . 3) The direct sum of sheaves is a sheaf.)*

Problem 1.4. Let R be a commutative ring, and $\mathfrak{p} \subseteq R$ a prime. Then

$$k(\mathfrak{p}) = \frac{R_{\mathfrak{p}}}{\mathfrak{p}R_{\mathfrak{p}}}$$

is isomorphic to the field of fractions of R/\mathfrak{p} .

Problem 1.5. If $I, J \subseteq R$ are ideals, then $V(I) = V(J)$ if and only if $\sqrt{I} = \sqrt{J}$.

Problem 1.6. Show that $D(x_1, \dots, x_n) \subseteq \operatorname{spec} R$ is compact.

2. HOMEWORK 2: "DUE" 22 FEBRUARY

Problem 2.1. Let $R = k[x_1, \dots, x_n]$ with k algebraically closed.

- a) Prove that the following forms of the Nullstellensatz are equivalent.
- i) \mathcal{I} and \mathcal{Z} give a bijection between the algebraic sets of k^n and radical ideals of R .
 - ii) For every proper ideal $I \subseteq R$, we have

$$\sqrt{I} = \bigcap_{\mathfrak{m} \in V(I) \cap \operatorname{MaxSpec} R} \mathfrak{m}$$

- b) Use Lemma 4.8 to show that ii) holds in R . This provides a second proof of the nullstellensatz.

Problem 2.2. Assuming k is algebraically closed, show that every point in \mathbb{P}_k^n is an algebraic set.

Problem 2.3. For an algebraically closed field k , show that

$$\mathcal{O}_{\mathbb{P}_k^n}(\mathbb{P}_k^n) \cong k$$

Problem 2.4. Prove that every morphism of projective varieties is regular.

Problem 2.5. Let k be an algebraically closed field.

a) Suppose $I \subseteq k[x_1, \dots, x_n]$ and $J \subseteq k[y_1, \dots, y_m]$ are ideals such that

$$\frac{k[x_1, \dots, x_n]}{\sqrt{I}} \cong \frac{k[y_1, \dots, y_m]}{\sqrt{J}}.$$

Show that varieties $V(I)$ and $V(J)$ are isomorphic.

b) Conclude that for an affine variety V , and an open set $U \subseteq V$, the definition of a regular function

$$\sigma: U \rightarrow k$$

does not depend on the choice on the presentation of the coordinate ring of V .

c) Formulate the corresponding statement for quasi-projective varieties.

Problem 2.6. Let V be the image of the following map.

$$\varphi: \mathbb{P}_{\mathbb{C}}^1 \rightarrow \mathbb{P}_{\mathbb{C}}^3 \quad \varphi(x_0, x_1) = (x_0^3, x_0^2 x_1, x_0 x_1^2, x_1^3)$$

(1) Show that

$$V = V(y_0 y_3 - y_1 y_2, y_1^2 - y_0 y_2, y_2^2 - y_1 y_3).$$

(*Hint:* Restrict to the open affine sets $y_0 \neq 0$ and $y_3 \neq 0$.)

(2) Compute the corresponding map on coordinate rings

$$\hat{\varphi}: \mathbb{C}[V] \rightarrow \mathbb{C}[\mathbb{P}_{\mathbb{C}}^1].$$

(3) Show that φ is a homeomorphism, but not an isomorphism of varieties.

Problem 2.7 (Optional: for the benefit of the motivated student and will not be on the final exam). Let R be a commutative ring. Let $U \subseteq \text{spec } R$ be an open set. A function

$$\sigma: U \rightarrow \bigsqcup_{\mathfrak{p}} R_{\mathfrak{p}}$$

is regular, if there is an open cover $U = \bigcup U_i$ and elements $f_i, g_i \in R$ with $U_i \subseteq D(g_i)$ such that for every $\mathfrak{p} \in U_i$,

$$\sigma(\mathfrak{p}) = \frac{f_i}{g_i} \in R_{\mathfrak{p}}.$$

Let $\mathcal{O}_R(U)$ denote the collection of regular functions on U .

A sheaf which is isomorphic to the pair $(\text{spec } R, \mathcal{O}_R)$ is called an affine scheme. A scheme is a sheaf \mathcal{O} on a space X such that there is an open cover $U = \bigcup U_i$ where $(U_i, \mathcal{O}|_{U_i})$ is an affine scheme.

Show the following.

- \mathcal{O}_R is a sheaf of rings
- $\mathcal{O}_R(\text{spec } R) = R$
- $\mathcal{O}(D(g)) = R_g$
- For any $\mathfrak{p} \in \text{spec } R$, $\mathcal{O}_{R, \mathfrak{p}} = R_{\mathfrak{p}}$.
- Any affine variety over an algebraically closed field is an affine scheme.
- Any variety over an algebraically closed field is scheme.