

EXERCISES

1. HOMEWORK 1: "DUE" 8TH FEBRUARY

Problem 1.1. Let \mathcal{F} be a sheaf on a topological space X . Suppose that for every $\mathfrak{p} \in X$, $\mathcal{F}_{\mathfrak{p}} = 0$. Show that $\mathcal{F} = 0$.

Problem 1.2. Let $f: \mathcal{F} \rightarrow \mathcal{G}$ be a homomorphism of sheaves over X . (*Hint: use the previous problem*)

a) Show that the following are equivalent. This is the definition of an injective sheaf.

- i) $\ker f = 0$
- ii) f_U is injective for every open sets $U \subseteq X$
- iii) $f_{\mathfrak{p}}: \mathcal{F}_{\mathfrak{p}} \rightarrow \mathcal{G}_{\mathfrak{p}}$ is injective for every $\mathfrak{p} \in X$

b) Show that the following are equivalent. This is the definition of a surjective sheaf.

- i) $\operatorname{coker} f = 0$
- ii) $f_{\mathfrak{p}}: \mathcal{F}_{\mathfrak{p}} \rightarrow \mathcal{G}_{\mathfrak{p}}$ is surjective for every $\mathfrak{p} \in X$.

c) Show that the following are equivalent. This is the definition of a sheaf isomorphism.

- i) $\ker f = \operatorname{coker} f = 0$
- ii) f_U is an isomorphism for every open sets $U \subseteq X$
- iii) $f_{\mathfrak{p}}: \mathcal{F}_{\mathfrak{p}} \rightarrow \mathcal{G}_{\mathfrak{p}}$ is an isomorphism for every $\mathfrak{p} \in X$.
- iv) There exists a sheaf homomorphism $g: \mathcal{G} \rightarrow \mathcal{F}$ such that $g_U \circ f_U$ and $f_U \circ g_U$ are the identity on $\mathcal{G}(U)$ and $\mathcal{F}(U)$ respectively for all open sets $U \subseteq X$.

Problem 1.3. Give an example of a space X and a sheaf homomorphism $f: \mathcal{F} \rightarrow \mathcal{G}$ over X such that f is surjective but f_U is not surjective for some open set $U \subseteq X$. (*Hint: Use the following constructions. Let G be an Abelian group. 1) Assign every open set of \mathbb{R} to the group G . 2) Take a point $\mathfrak{p} \in X$. For any open set $U \subseteq \mathbb{R}$, assign G to U if $\mathfrak{p} \in U$. Otherwise assign 0 to U . This is called the sky scraper sheaf of G at \mathfrak{p} . 3) The direct sum of sheaves is a sheaf.)*

Problem 1.4. Let R be a commutative ring, and $\mathfrak{p} \subseteq R$ a prime. Then

$$k(\mathfrak{p}) = \frac{R_{\mathfrak{p}}}{\mathfrak{p}R_{\mathfrak{p}}}$$

is isomorphic to the field of fractions of R/\mathfrak{p} .

Problem 1.5. If $I, J \subseteq R$ are ideals, then $V(I) = V(J)$ if and only if $\sqrt{I} = \sqrt{J}$.

Problem 1.6. Show that $D(x_1, \dots, x_n) \subseteq \operatorname{spec} R$ is compact.