EXERCISES

1. Homework 1: "Due" 8th February

Problem 1.1. Let $\mathcal{F}$ be a sheaf on a topological space X. Suppose that for every $\mathfrak{p} \in X$, $\mathcal{F}_\mathfrak{p} = 0$. Show that $\mathcal{F} = 0$.

Problem 1.2. Let $f \colon \mathcal{F} \to \mathcal{G}$ be a homomorphism of sheaves over X. (*Hint: use the previous problem*)

a) Show that the following are equivalent. This is the definition of an injective sheaf.
   - i) $\ker f = 0$
   - ii) f_U is injective for every open sets $U \subseteq X$
   - iii) $f_\mathfrak{p} \colon \mathcal{F}_\mathfrak{p} \to \mathcal{G}_\mathfrak{p}$ is injective for every $\mathfrak{p} \in X$

b) Show that the following are equivalent. This is the definition of a surjective sheaf.
   - i) $\operatorname{coker} f = 0$
   - ii) $f_\mathfrak{p} \colon \mathcal{F}_\mathfrak{p} \to \mathcal{G}_\mathfrak{p}$ is surjective for every $\mathfrak{p} \in X$.

c) Show that the following are equivalent. This is the definition of a sheaf isomorphism.
   - i) $\ker f = \operatorname{coker} f = 0$
   - ii) f_U is an isomorphism for every open sets $U \subseteq X$
   - iii) $f_\mathfrak{p} \colon \mathcal{F}_\mathfrak{p} \to \mathcal{G}_\mathfrak{p}$ is an isomorphism for every $\mathfrak{p} \in X$.
   - iv) There exists a sheaf homomorphism $g \colon \mathcal{G} \to \mathcal{F}$ such that $g_U \circ f_U$ and $f_u \circ g_U$ are the identity on $\mathcal{G}(U)$ and $\mathcal{F}(U)$ respectively for all open sets $U \subseteq X$.

Problem 1.3. Give an example of a space X and a sheaf homomorphism $f \colon \mathcal{F} \to \mathcal{G}$ over X such that f is surjective but f_U is not surjective for some open set $U \subseteq X$.

Problem 1.4. Let R be a commutative ring, and $\mathfrak{p} \subseteq R$ a prime. Then

$$k(\mathfrak{p}) = \frac{R_\mathfrak{p}}{\mathfrak{p}R_\mathfrak{p}}$$

is isomorphic to the field of fractions of $R_\mathfrak{p}$.

Problem 1.5. If $I, J \subseteq R$ are ideals, then $V(I) = V(J)$ if and only if $\sqrt{I} = \sqrt{J}$.

Problem 1.6. Show that $D(x_1, \ldots, x_n) \subseteq \operatorname{spec} R$ is compact.